ON THE EQUILIBRIUM OF AN ELASTO-PLASTIC BODY SUBMITTED TO THE DEFORMATION BY TRACTION

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ABSTRACT. We formulate some problems modelling the local behaviour of a plastic material with hardening. We express the directional linear hardening in term of a hidden internal variable. In this context there exist an admissible loading intensity and a permissible displacement for a large deformation.

Introduction. We focus our attention on an elastic-plastic body submitted to the deformation by traction. Suppose that the material is governed by a nonlinear subdifferential constitutive law. We express the directional linear hardening in term of a hidden internal variable: the dimensionless plastic work. We formulate an equilibrium problem under restrictions on the hardening factor, on *Lagrange* multiplier of the constitutive law and in case that an intuitively condition on the direction of the forces is satisfied. We can derive a loading intensity and an admissible displacement in context of large deformation.

1. Some preliminaries considerations. We have considered a nonlinear elastic-plastic body, that occupies a region Ω in R^3, its boundary consisting in disjoint parts $\Gamma_u, \Gamma_t, \Gamma_c$, on Γ_u the displacement is given, on Γ_t the surface tractions are given, on Γ_c the unilateral conditions with Coulomb's friction are given. Many useful comments about subject can be find in [3], [5], [6]. The body is subjected to body forces f_0 over Ω and surface traction$\gamma\vec{\tau}$, for some traction vector with prescribed direction. Here and in the sequel $u = (u_1, u_2, u_3)$ is the displacement vector field, the superposed dot denotes time derivative, $\sigma = \{\sigma_{ij}\}_{i,j=1,2,3}$is the stress tensor field, $\bar{n}$ is the outward unit normal vector field on the surface Γ of the domain Ω, θ is a scalar time dependent function.

We will investigate the problem:

Problem 1.1 (*quasi-static case* [7]): *Find* (u, σ, θ^*) *in admissible vector valued spaces, such that*

$\dot{\varepsilon}(x,t) = A_\tau(u(x,t))\,\dot{u}(x,t)$, *for all* $(x,t) \in \Omega \times [0,T]$, $u(x,0) = \dot{u}(x,0) = 0$, *for all* $(x,t) \in \Gamma_u \times [0,T]$ (*geometric equation*);

$A^*_\tau(u(x,t))\,\sigma(x,t) = f_0(x,t)$, *for all* $(x,t) \in \Omega \times [0,T]$ and

$A^*_\tau(u(x,t))\,\sigma(x,t)\,.\bar{n}(x,t) = \gamma(x,t)\,\bar{\vartheta}$, *for all* $(x,t) \in \Gamma_t \times [0,T]$ (*equilibrium equation*);

$-R(x,t) \in \partial\Phi(\dot{u}(x,t))$, *for all* $(x,t) \in \Gamma_s \times [0,T]$,

$R(t,x) = A^*_\tau(u(x,t))\,\sigma(x,t)\,.\bar{n}(x,t)$, *for all* $(x,t) \in \Gamma_s \times [0,T]$ (*unilateral contact friction with Coulomb's law*);

$\left(\dot{\varepsilon}(x,t), -\dot{\theta}(t)\right) \in \partial\Psi^c(\sigma(x,.t), \theta^*(t))$, *with, for all* $(x,t) \in \Omega \times [0,T]$.

The dynamical case was intensively studied in [3], [4], [8], using technical approach by penalty method. Here we reconsider the dynamical problem in the context of nonlinear geometric strain-displacement relation and develop some considerations about existence and regularity solution. In [1] apart from the mechanical reasons over shrink fitted shaft, or bushing assemblage of brakes was formulated a frictional joint model of an elastic body and an static deformation are studied. In the sequel the boundary part Γ_c is the contact boundary where the body may came into contact with a rigid support. Any displacement and traction forces on Γ_c can be decomposed into normal and tangential components: $u_N = u_i n_i$ (summation convention is used), $\sigma_N = \sigma_{ij} n_i n_j$, $u_{Ti} = u_i - u_N n_i, i = 1,2,3$, $\sigma_{Ti} = u_i - u_N n_i$, $i = 1,2,3$.

We also introduce

Problem 1.2 (*dynamical nonlinear case*): *Find* (u, σ, θ^*) *in admissible vector valued spaces, such that*

$\ddot{u}(x,t) - A_\tau^*(u(x,t))\sigma(t,x) = f_0(x,t)$, *for all* $(x,t) \in Q^+ = \Omega \times I$ (*dynamical equilibrium equation*);

$\left(\dot{\varepsilon}(x,t), -\dot{\theta}(t)\right) \in \partial\Psi^c(\sigma(x,.t), \theta^*(t))$, *with* , (*nonlinear constitutive law*) *for all* $(x,t) \in \Omega \times [0,T]$, *particularly*

$\sigma(x,t) = A^1\varepsilon(u(x,t)) + A^2\dot{\varepsilon}(u(x,t))$ (nonlinear visco-elastic law),

$\dot{u}(x,t) = v(x,t)$, for all $(x,t) \in S_u^+ = \Gamma_u \times I$ (prescribed velocity);

$A_\tau^*\sigma(x,t) = f_1(x,t)$, for all $(x,t) \in S_t^+ = \Gamma_t \times I$ (*the traction forces*);

$\dot{u}_N(x,t) \leq 0, \quad A_\tau^*\sigma(x,t) \leq 0, \quad \dot{u}_N(x,t) A_\tau^*\sigma(x,t) = 0$, *for all* $(x,t) \in S_c^+ = \Gamma_c \times I$ (*unilateral contact*);

$$\dot{u}_T(x,t) = 0 \Rightarrow |\sigma_T(x,t)| \leq \nu |\sigma_N(x,t)|,$$

for $\dot{u}_T(x,t) \neq 0$, $\sigma_T(x,t) = -\nu|\sigma_N| \frac{\dot{u}_T}{|\dot{u}_T|}$, for all $(x,t) \in S_c^+$ (*Coulomb*'s *friction condition*);

$u(x,0) = \dot{u}(x,0) = 0$ (initial data).

2. Nonlinear geometric mapping and dual spaces for an elasto-plastic model. Let Ω be an open, bounded, connected subset of R^3 with a boundary Γ, a Lipschitz boundary is sufficient.

Consider $U = \left\{u : \Omega \times I \to R^3\right\}$ the space of admissible displacements, $\dot{U}$the space of admissible velocities, paired with F the space of admissible forces acting on the body by the bilinear form $(\dot{u}, \rho f)_e$, the density of external power. Let E be the strain *Green* space, Σ the dual space of E, that is the *Kirchhoff* stress space; for a large deformations we take $\dot{E}$ the admissible strain rate space, the dual pair of $\Psi^c(\tau, \phi^*) \geq \langle A_T(u)\dot{u}, \tau - \sigma\rangle + \left\langle -\dot{\theta}, \phi^* - \theta^*\right\rangle_I$ and $\sigma \in \Sigma$ is given by $(\dot{\varepsilon}, \sigma)_i = \dot{\varepsilon} : \sigma = Tr\left(\dot{\varepsilon}.\sigma^T\right) = \dot{\varepsilon}_{ij}\sigma_{ij}$, the density of internal power.

Let $\varepsilon : U \to E$ be the *Green - Saint - Venant* mapping, giving the strain tensor, $\varepsilon(v) = \frac{1}{2}\left(\nabla v + \nabla v^T + \nabla v.\nabla v^T\right)$, for all $v \in U$, where the nabla operator ∇ is defined from U to E. Note that $\frac{d\varepsilon}{dv}(u)$ is the directional derivative of ε at $u \in U$ in the direction v, the so called *tangent geometric mapping*. We denote $A_T(u)v = \frac{d\varepsilon}{dv}(u)$. It is a simple calculus to show

Lemma 2.1: The tangent geometric mapping is expressed by the relation

$$\frac{d\varepsilon}{d\nu}(u) = \frac{1}{2}\left[\nabla\nu + \nabla\nu^T + \nabla\nu\nabla u^T + \nabla u\nabla\nu^T\right].$$

We omit the demonstration, more precise the map $A_T(t,u)$ is exactly the material derivative of $\varepsilon(u)$, that is we have

Lemma 2.2: There exist $A_T(t,u)\dot{u} = \left[\left(I + \nabla u^T\right)\nabla\right]_{sym}\dot{u}$

By this result it mean that $A_T(t,u)$ is an affine map; we observe that its second derivative does not depend on u, in other words$\delta^2 A_T(t,u) = 0$.

In this way we emphasize the nonlinear map$A_T(u) : \dot{U} \to \dot{E}$,

$$A_T(t,u)\dot{v} = \frac{1}{2}\left[\nabla\dot{v} + \nabla\dot{v}^T + \nabla u\nabla\dot{v}^T + \nabla u^T\nabla\dot{v}\right],$$

Let $A_T^*(u)$ be the conjugate map of $A_T(u)$, defined by a Gaussian transformation

$\langle A_T(u)\dot{u}, \sigma\rangle_\Omega = (\dot{u}, A_T^*(u)\sigma)_{\bar{\Omega}}$, where $\langle .,.\rangle_\Omega = \int_\Omega (.,.)_i\,dx$ and $(.,.)_{\bar{\Omega}} = \int_\Omega (.,.)_e\,dx + \int_\Gamma (.,.)_e\,da$.

In order to understand the Gaussian transformation we write, taking into account the symmetric stress tensorσ,

$$\frac{1}{2}\int_{\Omega}\left\{\nabla\dot{u}+\nabla\dot{u}^{T}+\nabla u\nabla\dot{u}^{T}+\nabla\dot{u}\nabla u^{T}\right\}:\sigma dx=$$

$$=\int_{\Gamma}\left(I+\nabla u^{T}\right)\sigma\bar{n}da-\int_{\Omega}\left(I+\nabla u^{T}\right)\nabla\sigma.\dot{u}dx,$$

and identifying the terms we obtain

$$A_{T}^{*}\left(t,u\right):\Sigma\rightarrow F, A_{T}^{*}\left(t,u\right)\sigma=-\left(I+\nabla u^{T}\right)\nabla\sigma, in\ \Omega,$$

$$A_{T}^{*}\left(t,u\right)\sigma\bar{n}=-\left(I+\nabla u^{T}\right)\nabla\sigma.\bar{n}, on\ \Gamma, \text{where } \vec{n} \text{ is the outward normal on } \Gamma.$$

Remark 2.1: The expression $\left(I+\nabla u^{T}\right)\nabla\sigma$ is understood in the sense of trace space.

Under the additional conditions about the boundary, that is $\partial\Omega\in C^{\infty}$, denote $\Gamma_{fc}=\Gamma_{f}\cup\Gamma_{c}$ an open subset of $\partial\Omega$, $\partial\Gamma_{fc}\in C^{\infty}$, the space of traces on Γ_{fc} of the displacements $u\in$V $=\left\{v\in H^{1}\left(\Omega\right)/v=0_{o}n\Gamma_{u}\right\}$ is the space $H_{00}^{1/2}\left(\Gamma_{fc}\right)$, see [12], defined by

$$(2.1)\quad H_{00}^{1/2}\left(\Gamma_{fc}\right)=\left\{w\in H^{1/2}\left(\Gamma_{fc}\right)/\frac{1}{\sqrt{\rho}}w\in L^{2}\left(\Gamma_{fc}\right)\right\},$$

where $\rho\left(x\right)=d\left(x,\partial\Gamma_{fc}\right)$. Hence, the trace operator $\gamma:V\rightarrow H_{00}^{1/2}\left(\Gamma_{fc}\right)$ is linear, surjective and continuous when $H_{00}^{1/2}\left(\Gamma_{fc}\right)$ is equipped with its natural norm.

Take $X=V^{3}$ the space of vector valued functions, then the trace operator $\gamma:X\rightarrow H_{00}^{1/2}\left(\Gamma_{fc},R^{3}\right)=\left(H_{00}^{1/2}\left(\Gamma_{fc}\right)\right)^{3}$ is a linear, surjectiv and continuous, hence $H_{00}^{1/2}\left(\Gamma_{fc},R^{3}\right)$ is isomorphic to the quotient space$X/_{H_{0}^{1}(\Omega,R^{3})}$. Denote also $\tilde{X}=\left\{v\in X/\left(A_{T}^{*}\sigma\right)v\in L^{2}\left(\Omega,R^{3}\right)\right\}$, for $u\in\tilde{X}$ define the continuous linear form, given by $\mathrm{l}_{u}\left(v\right)=a\left(u,v\right)+\int_{\Omega}\left(A_{T}^{*}\sigma\left(u\right)\right)vdx$, where the bilinear form a is defined by $a\left(u,v\right)=\int_{\Omega}\varepsilon\left(u\right):\sigma\left(v\right)dx$. The functional l_{u} depends only upon the restriction $v/_{\Gamma_{fc}}$, hence can be identified with an element of $\left(H_{00}^{1/2}\left(\Gamma_{fc},R^{3}\right)\right)^{*}$, denoted $A_{T}^{*}\sigma\left(u\right).\bar{n}$ and given by the Green's formula:

(2.2) $\langle A_{T}^{*}\sigma\left(u\right).\bar{n},w\rangle=a\left(u,v\right)+\int_{\Omega}A_{T}^{*}\sigma\left(u\right)vdx$, for all$w\in H_{00}^{1/2}\left(\Gamma_{fc},R^{3}\right)$,

where $v\in X$ is any element such that $v/_{\Gamma_{fc}}=w$.

The last relation reveals that the mapping $u\in\tilde{X}\rightarrow A_{T}^{*}\sigma\left(u\right).\bar{n}\in\left(H_{00}^{1/2}\left(\Gamma_{fc},R^{3}\right)\right)^{*}$ is linear and continuous map. Since $n\in C^{\infty}\left(\partial\Omega,R^{3}\right)$, $\partial\Omega\in C^{\infty}$ and $A_{T}^{*}\sigma\left(u\right).\bar{n}$ is a distribution on Γ_{fc}, then $\sigma_{N}\left(u\right)=\left(A_{T}^{*}\sigma\left(u\right).\bar{n}\right).\bar{n}$ is well defined on Γ_{fc}, for all $u\in\tilde{X}$, and $\sigma_{N}\left(u\right)\in\left(H_{00}^{1/2}\left(\Gamma_{fc}\right)\right)^{*}$. Of course, $u\in\tilde{X}\rightarrow\sigma_{N}\left(u\right)\in\left(H_{00}^{1/2}\left(\Gamma_{fc}\right)\right)^{*}$ is linear and continuous, as is the map $u\in\tilde{X}\rightarrow\sigma_{T}\left(u\right)=A_{T}^{*}\sigma\left(u\right).\bar{n}-\sigma_{N}\left(u\right).\bar{n}\in H_{00}^{1/2}\left(\Gamma_{fc},R^{3}\right)$.

3. Unilateral contact conditions with friction. The contact of a body with a rigid support satisfies the classical law of *Signorini* type: $\sigma_{N}\leq0$, $u_{N}-g\leq0$, $\sigma_{N}\left(u_{N}-g\right)=0$ on Γ_{c}, where g is a positive number, measuring the initial gap between the body and the rigid support; there is no sign restriction for g.

In a more general setting, let $K\subset\ \ U$ be a subset of the admissible displacements of every boundary point of Γ_{c}, assume that there exists a real function $g:\ \ U\rightarrow R$, such that any satisfies $g\left(u\left(x,t\right)\right)\leq0$, for all time of loading $t\in I$ and $x\in\Gamma_{c}$. For any admissible u we associate the subset $V_{K}\left(u\right)\subset\dot{U}$ of admissible velocities defined by $v\in V_{K}\left(u\right)\Leftrightarrow g\left(u\right)+\ \ \alpha\nabla g\left(u\right)v\in$

$R_- = (-\infty, 0]$, for some $\alpha \in R_+$, see [2]. A generalized *Signorini-Fichera* conditions on Γ_c, expressing the contact with unilateral support, having a nonlinear intensity $g$is

(3.1) $g(u) \leq 0,\ R_N(u) \leq 0,\ g(u) R_N(u) = 0$, on Γ_c,

where $R_N(u) = \left(I + \nabla u^T\right) \nabla\sigma.\bar{n}$ is a traction vector on the part Γ_t of boundary Γ, as a reaction of the rigid obstacle.

The friction law is that of *Coulomb*, which can be written as $|\sigma_T| \leq \nu |\sigma_N|$, on Γ_c, that is, if $|\sigma_T| < \nu |\sigma_N|$, then $\dot{u}_T = 0$ and if $|\sigma_T| \equiv \nu |\sigma_N| > 0$, then $\dot{u}_T = -\lambda\sigma_T$, $\lambda \geq 0$, here ν is a positive function, named the friction coefficient.

Example 3.1: Let $g(u) = u_N - g$ be an affine intensity function, where $u_N = u_i n_i$, $\bar{n}$ is the outward normal to $\sigma_N(u) = \sigma_N(\bar{u})$, then the condition (3.1) reduces to the classical one. For such a body, our goal is to define the cone of limit velocity fields. A limit state of the mechanical system is a state in which a constant rigid velocity field can be superposed onto a quasi-static deformation. Let $Q = \{w / w_i = \alpha_i + \beta_{ij} x_j, \beta_{ij} = -\beta_{ji}\}$ be the set of rigid body velocities, α_i and $\beta_{ij}$are real constants. Suppose that a kinematic contact $u_N - g \leq 0$ must be satisfied both at the instance when the limit velocity added to the quasi-static deformation and when a such a motion has continued for some short time. For a given displacement u we therefore define the cone of limit velocities as

$$K^\infty(u) = \{w \in Q / u_N + \eta w_N \leq g, u_N \leq g,\ on\ \Gamma_c, (\forall)\, \eta > 0, w \neq 0\} =$$

$$= \{w \in Q / w_N \leq 0\ on\ \Gamma_c, w \neq 0\}.$$

Each $w \in K^\infty(u)$ divides Γ_c into two complementary parts $\Gamma_c^-(w) = \{x \in \Gamma_c / w_N(x) < 0\}$ and $\Gamma_c^0(w) = \{x \in \Gamma_c / w_N(x) = 0\}$

Consider a velocity $w \in K^\infty(u)$, that is added to a quasi-static deformation at initial time, $t = 0$, then for $t > 0$ and $x \in \Gamma_c^-$ $(w(x) < 0)$ we see from the contact law relation that $\sigma_N(x) = 0$ and from *Coulomb* friction condition we have $\sigma_T(x) = 0$. Thus, the quasi-static deformation of the body at the limit state generated by w suppose the contact and friction conditions only on $\Gamma_c^0(w)$. Suppose $w = \gamma\bar{t}$, where $\bar{t}$ is a preserved direction, along which act the traction forces, $\Gamma_c^0(w)$ is an open set and each of its connected parts will be part of cylindrical or plane surfaces parallel with the $\bar{t}$- direction.

Note that for a chosen w the problem can be interpreted, at least for a flat $\Gamma_c^0(w)$, as a steady sliding problem. This is a case when no rotations are involved, i. e. $\beta_{ij} = 0$. If $\beta_{ij} \neq 0$, this interpretation is not possible due to the incapability to model large rotations.

Substituting the displacement $u = u^0 + \gamma\bar{t}$in the constitutive law, for example for a nonlinear elastic body, we obtain $\sigma_{ij} = E_{ijkl}\frac{\partial u_k^0}{\partial x_l}$ in $f_1 \in$. From the contact conditions we get $\sigma_N \leq 0$, $u_N^0 - g \leq 0$, $\sigma_N\left(u_N^0 - g\right) = 0$ on $\Gamma_c^0(w)$ and the friction law implies $\sigma_T = -\nu|\sigma_N|\,\bar{t}$ on $\Gamma_c^0(w)$.

The problem of finding the fields σ and u^0, such that

(3.2) $A_T^*(u)\,\sigma = \rho f_0$, $\sigma_{ij} = E_{ijkl}\frac{\partial u_k^0}{\partial x_l}$ in $f_1 \in$, $A_\tau^* \sigma(u)\,\sigma.\bar{n} = f_1(x)$ on Γ_t, $\sigma_N \leq 0$, $u_N^0 - g \leq 0$, $\sigma_N\left(u_N^0 - g\right) = 0$ on $\Gamma_c^0(w)$, $\sigma_T = -\nu|\sigma_N|\,\bar{t}$ on $\Gamma_c^0(w)$, constitutes a limit quasi-static problem.

Now define $\Phi_N^* : R \to R$, $\Phi_N^*(\tau_N) = \begin{cases} 0, \tau_N < 0 \\ +\infty,_o therwise \end{cases}$, take the conjugate function $\Phi(g(u)) = \sup\limits_{\tau_N \in N} \{(g(u), \tau_N) - \Phi_N^*(\tau_N)\} = \sup\limits_{\tau_N \in N_0} (g(u), \tau_N)$, where N_0 is a negative cone of $N = \{\tau_N / \tau_N = (\tau.\bar{n})\,\bar{n}, \tau \in \Sigma\}$, N is the subset of admissible boundary tractions. The contact conditions (3.1) can be summarized in the inclusion form

$$R_N(u) \in \partial\Phi_N(g(u)) = \psi_{N_0}(\tau_N) = \begin{cases} 0,_i f_g(u) \leq 0 \\ +\infty,_o therwise \end{cases},$$

where ψ_A is a characteristic function of a real subset A.

Let $\Phi_T(\dot{u}_T) = \int\limits_{\Gamma} \nu |R_N| |\dot{u}_T| da$ be the functional of an external power developed on the boundary Γ and its conjugate functional

$\Phi_T^*(\tau_T) = \left\{ \begin{array}{l} 0, if\, |\tau_T| \leq \nu |R_T| \\ +\infty, otherwise \end{array} \right. = \psi_{F_1}(\tau_T)$, here F_1 is a cone in F. The *Coulomb*'s friction law is equivalent with $-R_T \in \partial \Phi_T(\dot{u})$.

The behavior of the body on the part Γ_c of the boundary, having both unilateral contact conditions and *Coulomb*'s friction law satisfied, is characterized by the variational inequality:

Problem 3.1: Find $\dot{u}(t) \in V_K(u(t))$, such that

$R(u(t))(v - \dot{u}(t)) + \nu |R_N(u(t))| (|v_T| - \dot{u}_T(t)) \geq 0$, for all $v \in V_K(u(t))$, a. e. $t \in I$, $u(0) \in K$.

Remark 3.2: The last relation is equivalent to inclusion form $-R(u(t)) \in \partial \Phi(\dot{u}(t))$, for all $t \in I$, where $\Phi(v(t)) = \int\limits_{\Gamma_c} \nu(x) |R_N((v(x,t))) |v_T(x,t)|| da$.

4. Nonlinear constitutive law and the *Drucker*'s principle for an elastic-plastic body. Suppose $\sigma_c \in R_+$ a material constant, say an yield threshold at an experimental deformation by traction. We have in mind an old model of [7].

Definition 4.1: Denote by $\theta(t) = \frac{1}{\mu(\bar{\Omega})\sigma_c} \int\limits_{t \in I} \langle \sigma, A_T^*(u)\dot{u} \rangle_{\Omega} dt$ a dimensionless plastic power of the body, occupyingΩ, as an internal variable of a quasi-static state.

We take θ as a hardening factor and let Θ the space of admissible hardening factors of the body. Let θ^* be the conjugate function of $\theta \in \Theta$: $\theta^*(s) = \sup\limits_{t \in I} \{ts - \theta(t)\}$. We introduce the *linear hardening* property of the material.

Definition 4.2: The body is a material with linear hardening if $\theta^* = H\theta$, where H is a positive constant.

Suppose T a convex, lower semicontinuous function of the stress tensor, the plastic yield function η depending on the hardening factor, as a monotone function and construct $\varphi(\sigma, \theta^*) = T(\sigma) - \eta(\theta^*) - \sigma_c$.

Take $K \quad = \{(\sigma, \theta^*) \in \Sigma \times \Theta^* / \varphi(\sigma, \theta^*) \leq 0_i n_\Omega\}$, a closed convex set in $\Sigma \times \Theta^*$.

Example 4.1: Set $T(\sigma) = \langle \sigma, \sigma \rangle = \sum\limits_{i,j=1}^{3} \sigma_{ij}\sigma_{ij} (= \sigma_1^2 + \sigma_2^2 + \sigma_3^2$, after a diagonalization of the tensor σ), take $\eta = 0$ and $\sigma_c > 0$, we obtain the *Von Mises* condition.

We introduce the indicator function of the set

K, $\psi_K(\sigma, \theta^*) = \left\{ \begin{array}{l} 0, (\sigma, \theta^*) \in K \\ +\infty, otherwise \end{array} \right.$, it is a convex, lower semi-continuous and differentiable functional, ψ_K is so called complementary plastic super-potential, denoted Ψ^c. In this way, $\partial \Psi^c(\sigma, \theta^*)$ is a convex subset of $\dot{E} \times \Theta$.

Suppose that the two spaces implying the hardening factor are equal, $\Theta = \Theta^* = L^2(0, T; R)$, with dual pairing $\langle \xi, \varsigma \rangle_I = \int\limits_{t \in I} \xi(t) \varsigma(t) dt$. Define the duality between $\dot{E} \times \Theta$ and $\Sigma \times \Theta^*$ given by $\left\langle (\sigma, \theta^*), \left(\dot{\varepsilon}, -\dot{\theta}\right) \right\rangle = (\sigma, A_T(u)\dot{u})_i + \left\langle \theta^*, -\dot{\theta} \right\rangle_I$, which suggests, in a great generality, a constitutive relation of the hardening elastic-plastic material. Consider ψ_K^* the support function of the convex subset K, $\psi_K^*\left(\dot{\varepsilon}, -\dot{\theta}\right) = \sup\limits_{(\sigma, \theta^*) \in \Sigma \times \Theta^*} \left\{ \langle \sigma, \theta^* \rangle, \left(\dot{\varepsilon}, -\dot{\theta}\right) - \psi_K(\sigma, \theta^*) \right\} = \sup\limits_{(\sigma, \theta^*) \in \Sigma \times \Theta^*} \left\{ \langle \sigma, A_T(u)\dot{u} \rangle + \left\langle \dot{\theta}^*, \theta \right\rangle_I - \theta(T)\theta^*(T) \right\}$, where

$I = [0, T]$, therefore

$$(4.1)\ \left(\dot{\varepsilon}, -\dot{\theta}\right) \in \partial\Psi^c(\sigma,\theta^*) = \begin{cases} \left(\dot{\lambda}\frac{\partial\varphi}{\partial\sigma}, \dot{\lambda}\frac{\partial\varphi}{\partial\theta^*}\right),_i f_\varphi(\sigma,\theta^*) < 0 \\ (0,0),_\varphi(\sigma,\theta^*) = 0 \\ +\infty,_\varphi(\sigma,\theta^*) > 0 \end{cases}.$$

Then, for a given displacement $u \in U$, identifying each member of the dual pair, we obtain $\dot{\varepsilon} = A_T(u)\dot{u} = \dot{\lambda}\frac{\partial}{\partial\sigma}T(\sigma)$, $-\dot{\theta} = \dot{\lambda}\eta'(\theta^*)$, iff the constraints $\varphi(\sigma,\theta^*) < 0$, $\dot{\lambda} > 0$ are satisfied. Following some considerations of convex analysis, we can write

$$\langle A_T(u)\dot{u}, \tau - \sigma\rangle + \left\langle -\dot{\theta}(t), \phi^* - \theta^*\right\rangle_I \leq \Psi^c(\tau,\phi^*) - \Psi^c(\sigma,\theta^*), (\forall)(\tau,\phi^*) \in \Sigma \times \Theta^*.$$

Having in mind that (σ,θ^*) must be on the yield surface, take $\varphi(\sigma,\theta^*) = 0$ and the *Lagrange* multiplier $\dot{\lambda}$ is considered positive, we have a generalized *Drucker*'s postulate:

$$(4.2)\ \Psi^c(\tau,\phi^*) \geq \langle A_T(u)\dot{u}, \tau - \sigma\rangle + \left\langle -\dot{\theta}, \phi^* - \theta^*\right\rangle_I, \text{ for all } (\tau,\phi^*) \in \Sigma \times \Theta^*.$$

The request of satisfying the *Drucker*'s postulate can be formulate as a problem on the constraint K.

Problem 4.1: *Find* $(\sigma,\theta^*) \in \quad K$ *such that,*
$\langle A_T(u)\dot{u}, \tau - \sigma\rangle + \langle\theta, \phi^* - \theta^*\rangle_I \leq (\phi^*(T) - \theta^*(T))\theta(T)$, for all $(\tau,\phi^*) \in \quad K$.

5. Some variational formulation for a limit state and decoupled variational formulation. The domainΩ is a bidimensional subset. Define the bilinear form for an elastic behaviour of the body, $a(u,v) = \int\limits_\Omega \sigma_{ij} : \varepsilon_{ij}dx = \int\limits_\Omega E_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial v_k}{\partial x_l}dx$, the following Green's formula holds for a limit state

$$a\left(u^0, v\right) = -\int\limits_\Omega \frac{\partial\sigma_{ij}\left(u^0\right)}{\partial x_j}vdx + \int\limits_\Gamma \sigma_{ij}n_jv_ida.$$

Let V be a space of sufficiently smooth displacements defined on the closure of Ω, a convex set of admissible displacements is defined as

$K = \{u \in V/u_N - g \leq 0 \ on\ \Gamma_c\}$, we have in view the initial gap defined by means of a function $G = (G_1, G_2) \in V$, such that $g = G_N = G_in_{i|\Gamma_c}$, then we may write $K = g + K_0$, where $K_0 = \{v \in V/v_N < 0 \ on\ \Gamma_c\}$ is a convex cone with vertex in the origin. The complementarity conditions of contact law can be expressed as a variational inequality for a limit state u^0.

Problem 5.1: *Find* $u^0 \in K$, $\int\limits_{\Gamma_c} \sigma_N\left(v_N - u_N^0\right)da \geq 0, (\forall)\, v \in K.$

The variational formulation of the quasi-static problem of Signorini type is the following

Problem 5.2: *Find* $u^0(t) \in$K, *such that*
$a\left(u^0, v - u^0(t)\right) - \int\limits_{\Gamma_c} \nu\sigma_N\left(u^0\right)\left(v_T - u_T^0(t)\right)da \geq 0$, *for all* $v \in$K *and* $t \in I$.

Equivalently, in terms of K, we have

Problem 5.3: *Find* $\widehat{u} \in K_0$, *such that*
$(5.1)\ a\left(\widehat{u}, v - \widehat{u}(t)\right) - \int\limits_{\Gamma_c} \nu\sigma_N\left(\widehat{u}\right)\left(v_T - \widehat{u}_T(t)\right)da \geq$
$\geq \left(F.v - \widehat{u}(t)\right) + \left\langle l_g, v - \widehat{u}(t)\right\rangle - \left\langle g, v - \widehat{u}(t)\right\rangle,$
where $\sigma_N\left(u^0\right)$, $\sigma_N\left(\widehat{u}\right)$, $\sigma_N(g)$ *are stress tensors; the two solutions* u^0 *and* $\widehat{u}$ *are related as* $u^0 = \widehat{u} + g$, *here* $\langle F, v\rangle = \int\limits_\Omega f_0vdx + \int\limits_\Gamma \gamma t_iv_i$; $\langle l_g, v\rangle = \int\limits_{\Gamma_c} \nu\sigma_N(g)\,vda$ *and the forces* l_g *may be interpreted as being due to the shrink-fitting, if* $g < 0$.

We decompose the variational inequality from (5.1) into a variational inequality, related to the displacement and an equation related to the load multiplier. Assume that $L =$ Q$\cap$ K$_0$ is a linear subspace of V, then $v \in L$ has the property that $v_N = 0$ on Γ_c, that is $K^\infty(u) \subset L - \{0\}$: $L = \{v \in V/v = k\bar{t},_f or_some_k \in R\} = R.\bar{t}$, a one-dimensional subspace of V. We set $L^{\perp} = \left\{ v \in V / \int\limits_{\Omega} v.\bar{t} dx = 0 \right\}$, so that $V = \quad L \oplus \quad L^{\perp}$, where the orthogonality is made in the L^2-sense. Each element $u \in V$ may then be decomposed as $u = \bar{u} + r\bar{t}$, with $\bar{u} = \left(u.\bar{t} - \frac{1}{\mu(\Omega)} \int\limits_{\Omega} u\bar{t} dx \right) \bar{t} + u_s \bar{s}$, where $\bar{t} \perp \bar{s}$ then $\bar{u} \in L^{\perp}$ and $r = \frac{1}{\mu(\Omega)} \int\limits_{\Omega} u\bar{t} dx$. We now substitute the fields $u = \bar{u} + r\bar{t}$, $v = \bar{v} + s\bar{t}$, where $\bar{u}, \bar{v} \in \mathrm{L}^{\perp}$, into the variational problem. Since $a(u,v) = a(\bar{u}, \bar{v})$, $\sigma_N(u) = \sigma_N(\bar{u})$, one then obtains

$$a(\bar{u}, \bar{v} - \bar{u}) - \int\limits_{\Gamma_c} \nu \sigma_N(\bar{u})(v_T - u_T) da - (s - r) \int\limits_{\Gamma_c} \nu \sigma_N(\bar{u}) da \geq$$

$$\geq (F, \bar{v} - \bar{u}) + (s - r) \langle F, \bar{t} \rangle + \langle l_g, \bar{v} - \bar{u} \rangle + (s - r)(l_g, \bar{t}) - a(g, \bar{v} - \bar{u}).$$

A solution of the last variational inequality can be constructed from a solution of the following system:

Problem 5.4: *Find* $\bar{u} \in \mathrm{K}_0 \cap \mathrm{L}^{\perp}$, *such that* $-\int \nu \sigma_N(\bar{u})(v_T - u_T) da = (F, \bar{t}) + \langle l_g, \bar{t} \rangle$ *and for all* $\bar{v} \in \mathrm{K} \cap \mathrm{L}^{\perp}$,

$$a(\bar{u}, \bar{v} - \bar{u}) - \int\limits_{\Gamma_c} \nu \sigma_N(\bar{u})(v_T - u_T) da \geq (F, \bar{v} - \bar{u}) + (l_g, \bar{v} - \bar{u}) - a(g, \bar{v} - \bar{u}).$$

The first equation expresses global equilibrium in the $\bar{t}-$direction. A certain non-uniqueness of solutions has appeared. The problem is indifferent with respect to a rigid body displacement in the $\bar{t}-$direction.

6. A gap function and its compensatory operator. Let S_a be a time independent statically admissible set, defined by

$$\mathrm{S}_a \quad = \{(v, \tau, \phi^*) \in U \times \Sigma \times \Theta^* / A_T^*(v) \tau = f_0 \; in \; \Omega, A_T^*(v) \tau.\bar{n} = \gamma(\tau, v).\bar{t} \; on \; \Gamma_t\},$$

where $\gamma(\tau, v)$ is a *loading intensity* in a hardening deformation theory. Take $u(t) = v + \delta u(t)$, $\theta^*(t) = \phi^* + \delta\theta^*(t)$, assuming that v and ϕ^* are time independent and observe that $\delta\dot{u} = (\delta_u)'_t ,_\delta \dot{\theta}^* = (\delta\theta^*)'_t$, we make a simple calculus of the *Gateaux* derivative using the strain rate tensor $\varepsilon(v + \delta u)$, so we have

$$\frac{d}{dt} \varepsilon(u) = A_T(v + \delta_u)(v + \delta_u)'$$

$$= \frac{1}{2} \left[\nabla (\delta_u)' + \nabla \left(\delta_u^t\right)' + \nabla (\delta_u)' \nabla (u + \delta_u)^t + \nabla \left(\delta_u^t\right)' \nabla (v + \delta_u) \right] =$$

$$= A_T(v)(\delta_u)' + A_n(\delta_u)(\delta_u)'.$$

In this way we introduce the so called *compensatory operator* $A_n(u) : \dot{E} \to \dot{E}$, which is symmetric. Based on the same idea we can deduce $\frac{d}{dt} \left(\nabla(\delta_u) \nabla(\delta_u)^T \right) = 2A_n(\delta_u)(\delta_u)$ and $2 \int_0^{t_f} \left(A_n(\delta_u)(\delta_u)', S \right)_{\Omega} dt =$
$\int_0^{t_f} \frac{d}{dt} \left(\nabla \delta_u \nabla (\delta_u)^t, S \right)_{\Omega} dt = \int\limits_{\Omega} \nabla(\delta_u) \nabla(\delta_u)^T : S dx$, where S is a symmetric tensor with vanish initial state. We introduce $G(\delta_u, S) = \int_{\Omega} \nabla \delta_u \nabla (\delta_u)^t : S dx = G(u - v, S)$, the *gap function* associated with compensatory operator.

7. Some main results of existence, uniqueness and regularity. Let $H^1(I, U) = \left\{ u \in L^2(I, U) / \dot{u} \in L^2(I, U) \right\}$ be a Hilbert space, where U is too a *Hilbert* space, each element $u \in H^1(I, U)$ is called generic the *displacement process*, associated with an mechanical system. We can write the same for all spaces involved in this model: U, F, E, Σ. We have

Theorem 7.1: Suppose that $f_0 \in L^2\left(I,\left(H^1(\Omega)\right)^3\right)$, $\gamma \in L^\infty\left(I, H^1(\Gamma)\right)$, Φ is a convex, lower semi-continuous functional, $\Psi^c = \psi_S$, for some surface $S \subset H^1(I, \Sigma \times \Theta^*)$, H is a bounded function, then there exists a solution $(u, \sigma, \varepsilon, \theta) \in H^1(I, U \times \Sigma \times E \times \Theta^*)$ of the *Problem* 1.1

The uniqueness of the solution is a right consequence of the maximal monotony of the map $\partial\Psi^c$.

We can reduce the *Problem* 1.1 at the generalized sweeping process, as we have proved in [7] and we make here the same remarks about the deformation with hardening as in previous paper.

We return to the *Problem* 1.2 and let

$$C = \left\{v \in L^2\left(I, H^1(\Omega)\right)/v = \bar{u} \text{ on } S_u^+, v_n \leq 0 \text{ on } S_c^+\right\}$$

be the set of admissible functions, where $S = I \times \Gamma$, the problem reads

Problem 7.1: *Find* $\dot{u} \in C \cap B_0\left(I, L^2(\Omega)\right)$, with $u(x,0) = \dot{u}(x.0) = 0$ *in* Ω, *such that*

$$\int_{Q^+} \left\{\ddot{u}_i(v_i - \dot{u}_i) + \tilde{a}(u, v - \dot{u})\right\} dxdt + \int_{S_c} \nu\left|\sigma_n(u)\right|\left(|v_t| - |\dot{u}_t|\right) dxdt \geq$$

$\geq \int f_{0i}(v_i - \dot{u}_i)\, dxdt$, *for all* $v \in C$, *where* $\tilde{a}(v, w) = \sigma_{ij}(v)\,\varepsilon_{ij}(w)$, B_0 *the vector valued space of bounded functions.*

A new set of admissible functions $\tilde{V} = \left\{v \in L^2\left(I, H^1(\Omega)\right)/v = 0 \text{ on } S_u^+\right\}$ permit us to replace contact conditions, prescribing the normal component of boundary stress $\sigma_n = -\frac{1}{\varepsilon}\left[\dot{u}_n\right]_+$. The penalized problem reads

Problem 7.2: *Find* $\dot{u}_\varepsilon \in (\bar{u} + V) \cap B_0\left(I, L^2(\Omega)\right)$, *with* $u_\varepsilon(0) = \dot{u}_\varepsilon(0) = 0$, *such that*

$$\int_{Q^+} \left\{(\ddot{u}_{\varepsilon i} - f_0)(v_i - \dot{u}_{\varepsilon i}) + \tilde{a}(u_\varepsilon, v - \dot{u}_\varepsilon)\right\} dxdt +$$

$$+ \int_{S_c^+} \frac{|\dot{u}_{\varepsilon n}|_+}{\varepsilon} \left\{(v_n - \dot{u}_{\varepsilon n}) + \nu\left(|v_T| - |\dot{u}_{\varepsilon T}|\right)\right\} dadt \geq 0, \text{ for all } v \in \tilde{V}.$$

The existence of a solution of this problem can be proved using Faedo-Galerkin method [8].

Proposition 7.1: Let ν have a compact support, $\Gamma \in C^{1,1}$ and $f_0 \in L^2(Q^+)$, $\bar{u} \in H^1(Q^+)$ be such that $C \neq \emptyset$, then there exists at least one solution u_ε, for all ε small. We have the following estimations

$\|\dot{u}_\varepsilon\|_{L^2(I,H^1(\Omega)) \cap B_0(I,L^2(\Omega))} \leq C, \quad \frac{1}{\varepsilon}\left\|[\dot{u}_{\varepsilon n}]_+\right\|_{L^2(S_c) \cap H^{\frac{1}{2}}(S_c)} \leq C, \quad \|\ddot{u}_\varepsilon\|_{L^2(Q^+)} \leq C$, for same C and uniformly in ε.

Using the compact embedding theorems for Sobolev spaces and passing to the limit $\varepsilon \to 0$ we have

Theorem 7.2: Suppose that $f_0 \in L^2\left(I, \left(H^1(\Omega), R^3\right)\right)$, $v \in L^\infty\left(I, H^1(\Gamma_u)\right)$, A^1, A^2, $\nu \in L^\infty(\Omega)$, $f_1 \in L^\infty\left(I, H^1\left(\Gamma_t, R^3\right)\right)$, then there exists a unique solution $(u, \sigma, \varepsilon, \theta) \in H^1(I, U \times \Sigma \times E \times \Theta)$, with $\dot{u} \in L^\infty\left(I, \dot{E}\right)$, of the *Problem* 1.2.

In [7] a quasi-static process can be regarded as a limit process of dynamical one, presenting a small inertial term.

8. An application to the traction of a control sample, experimental results. A control sample is submitted to large deformation by traction, for which, by technological point of view, we are interesting to know some proper values of the internal stress. The mechanical system occupies a compact interval of real line and it is submitted to traction forces acting on the side b. Let $f(x,t)$ be a uni-dimensional density, continuous in $x \in [a,b]$ and Lipschitz continuous in $t \in [0,T]$.

We consider the displacement $u : [a,b] \times [0,T] \to R$, a function with bounded variation on $[a,b]$, Lipschitz continuous in t, satisfying the boundary conditions: $u(t,a) = 0,_u(t,b) = h(t)$, here h is a Lipschitz function, having an initial condition: $u(0,x) = u_0(x)$.

Denote $S = C([a,b]; R)$ the real space of real continuous functions on $[a,b]$, $E = M([a,b], R)$ the space of measures on $[a,b]$, we define the mechanical element $(S, (.,.), E)$ endowed by the duality $(s,m) = \int\limits_{[a,b]} sdm$, for all $s \in S$, $m \in E$. Let C be the subspace of measures, each of them having vanish total sum, N the subspace of continuous constant functions on $[a,b]$. Denote $A \in \quad E$ a positive measure having a density $a(x)$ on $[a,b]$, which represent the elastic coefficient al the material. The space E is endowed with the inner product $(e,\tilde{e}) = \int eA\tilde{e} = \int\limits_{[a,b]} e(x)\,a(x)\,\tilde{e}(x)\,dx$, where $e(x)$ and $\tilde{e}(x)$ are the density of measures e and $\tilde{e}$. Suppose H a *Hilbert* space with the norm given by the scalar product . In H we can identify the admissible elastic strain with admissible elastic stress. We have in view e, p, d which signify elastic strain, plastic strain and visible strain of the system, $e,p,d \in H$. We consider the following problem:

Problem 8.1: *Find* $e,p,d \in E$, *such that* $d \in C + h, e \in N + F, e = P_S d$, *where* $\boldsymbol{S} = \{s/\alpha(x,t) \le s(x,t) \le \beta(x,t)\}$ *is the closed convex of the theory of plasticity for a system having a nonlinear constitutive law of Hencky's type.*

If we consider the potential functional of the projection operator P_S, that is

$$f(d) = \sup_{\alpha \le s \le \beta} \left\{ (d,s) - \frac{1}{2}\|d\|^2 \right\},$$

we can give another formulation of the constitutive law, $e \in \partial f(d)$. In [7] is shown that f is a regular functional except the endpoints of the interval $[\alpha,\beta]$ ..

Let $V = \left\{ u(.,t) : [a,b] \to R/u(t,a) = 0, u \text{ continuous on } [0,T], \underset{[a,b]}{V} u < \infty \right\}$ be the space of admissible displacements of the system, then we formulate the minimization problem

$$\inf_{v \in C(h)} \left\{ f\left(h - \frac{dv}{dx}\right) + \left(\frac{dv}{dx}, F\right) \right\}, \quad C(h) = \left\{ v \in V/h - \frac{dv}{dx} \in C \right\}.$$

We remark that the minimization problem is a static one, but the deformation process is a quasi-static one. Using a penalty method, the properties of the functional f, a relaxation algorithm with a super-relaxation factor $\omega = 1.2$, $\omega = 0.1$we can find the time dependence of the elongation as in *Figure* 8.1

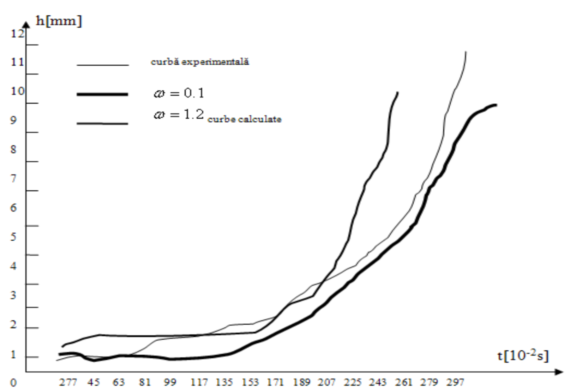

Fig. 8.1 –Dependenta de timp a alungirii: curb experimental; curbe calculate: ω=0,1; ω=1,2.

References

[1] L.E. Andersson & A. Klarbring, *On a class of limit states of frictional joints: formulation and existence theorem*, Quart. of Appl. Mathematics, vol. LV, no 1, 60-87, 1997

[2] A. Cimetiere, A. Leger & C. Naejus, *A functional framework for the Signorini problem with Coulomb's friction*, EDF, Directions des Etudes et Recherces, 1994

[3] O. Chen, M. Shillor & M. Sofonea, *Dynamic frictionless contact with adhesion*, Z. angew Math. Phys. 55, 32-47, 2004
[4] Y. Gao & G. Strang, *Geometric nonlinearity potential energy, complementary energy and the gap function*, Quart. of Appl. Math., vol XLVII, 487-504, 1989
[5] Y. Gao & T. Wierzbicky, *Bounding theorem in finite plasticity with hardening effect,* Quart. Appl. Math. vol XLVII, 395-403, 1989
[6] Y. Gao, *Dynamically loaded rigid-plastic analysis under large deformation*, Quart. Appl. Math., vol. XLVIII, 731-739, 1990
[7] C. Ghita, Metalurgie Matematica. *Analiza nestandard a proceselor metalurgice*, Ed. Academiei Romane, 1995
[8] J. Jarusek & C. Eck, *Dynamic contact problems with friction in linear viscoelasticity*, C.R. Acad. Sci. Paris, t. 322, S.1 497-502, 1996
[9] O. Klein, *Asymptotic behaviour for a phase-field model with hysteresis in one-dimensional thermo-visco-plasticity*, Appl. of Mathematics, vol. 49, no 4, 309-341, 2004
[10] J. J. Moreau, *On unilateral constraints, friction and plasticity, in New Variational Techniques in Mathematical Physics*, G. Capriz & G. Stampacchia eds., Ed. Cremonese, Roma, 1973
[11] P.D. Panagiotopoulos, *Multivalued Boundary Integral Equations for Inequality Problems. The convex case*, Acta Mechanica, vol. 70, 145-167, 1987
[12] P.J. Rabier & O.V. Savin, *Fixed Points of Multi-valued Maps and Static Coulomb Friction Problems*, Journal of Elasticity, 58, 155-176, 2000

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