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CONVEXITY AND LEAST SQUARE APPROXIMATION

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Abstract. In this paper we define the notion of *n-m convexity* and find the connection with *n-order convex function* defined by Tiberiu Popoviciu.

Introduction and notation

Let E be a set of real numbers that contains at least $m+1$ distinct points:
(1) $x_1, x_2, ..., x_{m+1}$
and $m+1$ real numbers:
(2) $y_1, y_2, ..., y_{m+1}$
We note with $V\left(x_{i_1}..., x_{i_{n+1}}\right)$ the Vandermonde's determinant on the nodes $x_{i_1}..., x_{i_{n+1}}$ and with $L\left(x_{i_1}, ...x_{i_{n+1}}; y_{i_1}, ..., y_{i_{n+1}}\right)$ the Lagrange polynomial on nodes $x_{i_1}..., x_{i_{n+1}}$ and corresponding values $y_{i_1}..., y_{i_{n+1}}$.

We call, as V. L. Gonciarov [1], for n¡m the interpolation polynomial of degree n in the meaning of least square determined by nodes (1) and numbers (2) the polynomial of degree at most n $Pn(x1, x2, ..., xm+1; y1, y2, ..., ym+1)$ that minimizes:

(3) $\sum\limits_{\text{k=1}}^{\text{m+1}} \left(P_n\left(x_1, x_2, ..., x_{m+1}; y_1, y_2, ..., y_{m+1}\right)(x_k) - y_k\right)^2$

Theorem 1. (V. L. Gonciarov [1]) *There is an unique polynomial of degree at most n that minimizes (3).*

We reproduce here the proof for using it in the next section of paper:

Proof: We will use the following notation:

(4) $s_k = \sum\limits_{i=1}^{m+1} x_i^k, k = 0, 1, ..., 2n \quad \gamma l = \sum\limits_{i=1}^{m+1} y_i x_i^l, l = 0, 1, ..., n..$

From the minimum condition of the sum (3) it result that the coefficients $a_i, i = 0, 1, ..., n$of the polynomial $Pn(x1, x2, ..., xm+1; y1, y2, ..., ym+1\ )$ satisfy the system:

$$\sum\nolimits_{i=0}^{n} a_i s_{i+k} = \gamma_k, k = 0, 1, ..., n.$$

It follows that:

$$(5)\ P_n\left(x_1, ...x_{m+1}; y_1, ...y_{m+1}\right)(x) = -\frac{\begin{vmatrix} c_0 & c_1 & ...c_n & \gamma_0 \\ c_1 & c_2 & ...c_{n+1} & \gamma_1 \\ ... & ... & ... & ... \\ c_n & c_{n+1} & ...c_{2n} & \gamma_n \\ 1 & x & ..x^n & 0 \end{vmatrix}}{\begin{vmatrix} c_0 & c_1 & ...c_n & \\ c_1 & c_2 & ...c_{n+1} & \\ ... & ... & ... & ... \\ c_n & c_{n+1} & ...c_{2n} & \end{vmatrix}},$$

where the denominator is (see [2]) is a sum of square of Vandermonde's determinants on $n+1$ distinct nodes from (1).

2. Main results

The connection between the polynomial $Pn(x1, x2, ..., xm{+}1; y1, y2, ..., ym{+}1\ )$ and Lagrange polynomials on nodes (1) is:

Theorem 2 *The polynomial* $Pn(x1, x2, ..., xm + 1; y1, y2, ..., ym + 1)$ *is a convex sum of the Lagrange polynomiasl on n+1 distinct nodes from (1):*

(6)
$$P_n(x_1, x_2, \ldots, x_{m+1}; y_1, y_2, \ldots, y_{m+1})(x) = \\ = \frac{\sum\limits_{1\le t_1\le t_2\le\ldots\le t_{n+1}\le m+1} V^2(x_{t_1}, x_{t_2}, \ldots, x_{t_{n+1}})L(P_n; x_{t_1}, x_{t_2}, \ldots, x_{t_{n+1}}; y_{t_1}, y_{t_2}, \ldots, y_{t_{n+1}})(x)}{\sum\limits_{1\le t_1\le t_2\le\ldots\le t_{n+1}\le m+1} V^2(x_{t_1}, x_{t_2}, \ldots, x_{t_{n+1}})}$$

Proof. The determinant that appears at the denominator in (5) is the determinant of the matrix product $B.C$ where:

$$B = -\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 0 \\ x_1 & x_2 & x_3 & \ldots & x_{m+1} & 0 \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_{m+1}^2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & x_3^n & \ldots & x_{m+1}^n & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n & y_1 \\ 1 & x_2 & x_2^2 & \ldots & x_2^n & y_2 \\ 1 & x_3 & x_3^2 & \ldots & x_3^n & y_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_{m+1} & x_{m+1}^2 & \ldots & x_{m+1}^n & y_{m+1} \\ 1 & x & x^2 & \ldots & x^n & 0 \end{pmatrix}$$

Using Cauchy-Binet formula (see [2]) it result that:

$$\det(BC) = \sum_{1\le i_1\le i_2\le\ldots\le i_{n+1}\le m+1} \det\left([b_{k,i_k}]_{k=\overline{1,n+2}}\right) \det\left([c_{k,i_k}]_{k=\overline{1,n+2}}\right)$$

If in the above sum $i_{n+2} \neq m+2$ the first determinant has a zero row and is null. If $i_{n+2} = m+2$ than the fisrt determinant is $V\left(x_{i_1}..., x_{i_{n+1}}\right)$ and the second determinant is $V\left(x_{i_1}, ..., x_{i_{n+1}}\right) L\left(x_{i_1}, ...x_{i_{n+1}}; y_{i_1}, ..., y_{i_{n+1}}\right)(x)$ so that the denominator from (5) is:

$$\sum_{1\le i_1\le i_2\le\ldots\le i_{n+1}\le m+1} V^2(x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1}})\cdot \\ \cdot L(P_n; x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1}}; y_{i_1}, y_{i_2}, \ldots, y_{i_{n+1}})(x)$$

and the numerator is (see the proof of theorem 1) :

$$\sum_{1\le i_1\le i_2\le\ldots\le i_{n+1}\le m+1} V^2(x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1}}).$$

For a function $f : E \to R$and $yi = f(xi)$ we denote the above polynomials with:

$$Pn(x1, x2, ..., xm + 1; f).$$

Definition 1 *The coefficient of xnfrom* $Pn(x1, x2, ..., xm + 1;$ *f) is called divided difference of order n-m of the function f on nodes (1) and will be denoted with:*

$$[x_1, ...x_{m+1}; f]_n$$

Definition 2 *The function $f: E \to R$ is called n - m order convex (respectively nonconcave,polynomial, nonconvex, concave) if for all distinct points (1) from E :*

$$[x1, x2, ..., xm + 1; f]n > (respectively \geq, =, \leq, <)0.$$

Theorem 2 *If the function $f: E \to R$ is a n-order convex function on E then f is n-m order convex on E for every $m > n + 1$.*

Proof. This theorem results from formula (6).

Theorem.3 *If $f: [a, b] \to R$ has continuous derivative of order $n + 1$ and f is n-m order convex on [a, b] then f is n-order convex on [a, b] .*

Proof. We suppose that f is not convex of order n . Than there is a point $c \in (a, b)$ in so that the derivative of order $n+1$ is negative in this point. The there is a neighbourhood of the point c so that the divided difference on every $n+2$distinct points from this neighbourhood is negative (the mean theorem for divided differences). If we choose all nodes (1) in this neighbourhood it results from (6) that f is not a *n-m order convex function.*

References

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[2] Samuel Karlin ¸Si William J. Studden. Total Positivity. Interscience Publisher, a division of John Wiley & Sons, New York.London.Sydney, 1967.

[3] Tiberiu Popoviciu. *Notes sur les fonctions convexes d'order supérieur IX*. Bull. Math. de la Soc. Roum. des Sci., LXIII(12), pp: 85—141, 1941.

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