THE SOLVING OF SOME STOCHASTIC DIFFERENTIAL EQUATIONS THAT INFLUENCES PERIODICAL MOVEMENTS

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ABSTRACT. Periodic or among-periodic variations in time and space of a dynamic system parameters, also know as oscillations, have an important role in the study of random phenomena with a certain degree of periodicity, phenomena that appear in all classes of science: physics, chemistry, economy, sociology, weather prediction, politics telluric science. The oscillations may be mechanical, electromechanical, electro temperature, temperature, electrical, seismically, of nebulosity. In this paper, there are some stochastically differential equations solved using Ito's formula, equations that can influence phenomena caused by periodic or among-periodic variation of random parameters.

Application 1: In the case when a point is solicited by two perpendicular random movements then the point is forced into a move on a ellipse-like way, and the equations are in the form

$$x = a\sin(\omega\, t + \varphi_1)\,,\; y = b\cos(\omega\, t + \varphi_2)$$

We can consider that the phase of the random move is a stochastic process modulated by a Brownian 1dimensional move obtaining a Brownian move on an ellipse

$$\left\{(y_1, y_2), \frac{y_1^2 a^2}{+}\frac{y_2^2 b^2}{=}1\right\} \tag{1}$$

The stochastic process $y_t = \{y_1(t), y_2(t)\}\,,\; y_1(t) = a\cos B_t, y_2(t) = b\sin B_t$ where B_t is a 1dimensional Brownian movement, the Brownian movement on an ellipse it's the solution for the stochastically equation

$$dy_t = -1/2 y_t d_t + M y_t dB_t,\; where\; M = \begin{pmatrix} 0 & -a/b \\ b/a & 0 \end{pmatrix}. \tag{2}$$

Demonstration: We will apply Itô's formula in order to calculate the stochastically differential, successively to the functions, [13]

$y_1(t) = f_1(t, B_t)\,,\; f_1(t,x) = a\cos x,\; x = B_t y_2(t) = f_2(t, B_t)\,,\; f_2(t,x) = b\sin x,\; x = B_t$

So the stochastically differential of f is

$$dy_i(t) = f_t' dt + f_x' dx + 1/2 f_{xx}'' dx^2,\; i \in \{1,2\} \tag{3}$$

As $dx^2 = dB_t^2 = dt$, [13], we obtain

$$dy_1(t) = -1/2 a\cos B_t dt - a\sin B_t dB_t, dy_2(t) = -1/2 b\sin B_t dt + b\cos B_t dB_t$$

Now we write in matrix-like form and we obtain:

$$\begin{pmatrix} dy_1(t) \\ dy_2(t) \end{pmatrix} = \begin{pmatrix} -a\sin B_t \\ b\cos B_t \end{pmatrix} dB_t - \frac{1}{2}\begin{pmatrix} a\cos B_t \\ b\sin B_t \end{pmatrix} dt =$$

$$= -\frac{1}{2}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 & -a/b \\ b/a & 0 \end{pmatrix}\begin{pmatrix} a\cos B_t \\ b\sin B_t \end{pmatrix} dB_t$$

and we note

$$(y_t) = \left(\begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right), \quad M = \left(\begin{array}{cc} 0 & -a/b \\ b/a & 0 \end{array} \right)$$

and we obtain equation (2).

Application 2: If the stochastically differential equation is like:

(4) $$dy_t = 1/2 y_t dt + y_t dB_t$$

Then the solution for the equations is

(5) $$y_t = e^{B_t}$$

Where $(B_t)_{t \in R}$is a 1dimensional Brownian movement?

Demonstration: We apply Itô's formula for stochastically differential calculation of the process: $y_t = f(t,x) = e^x$, $x = B_t$, calculating the partial derivations of the function $f(t,x) = e^x$, $f'_x(t,x) = e^x$, $f'_x(t,x) = e^x$,witch we replace with Itô's formula (3) and we obtain $dy_t = 0dt + e^{B_t} dB_t + 1/2 e^{B_t} dt$.

It mean the stochastically equation (4).

Application 3: Let be a differential equation like

(6) $$dy_t = -\frac{1}{t+1} y_t dt + \frac{1}{t+1} dB_t, \; y_0 = 0$$

Then the solution of the equation is

(6') $$y_t = \frac{B_t}{1+t}, \; B_0 = 0$$

Demonstration: In the same way, we apply Itô's formula for stochastically differential calculation of process $(y_t)_{t \in R}$ where $y_t = f(t, B_t)$, $y_t = f(t, B_t)$, and $f(t,x) = \frac{x}{1+t}$, $x = B_t$, calculating the partial derivates of the function

$$f(t,x) = \frac{x}{1+t},$$

we get

$$f'_t(t,x) = -\frac{x}{(1+t)^2}$$

$$f'_x(t,x) = \frac{1}{1+t}$$

$$f''_{xx}(t,x) = 0$$

which we replace with Itô's formula (3) and we obtain $dy_t = -\frac{B_t}{(1+t)^2} dt + \frac{1}{1+t} dB_t$, in fact the stochastically differential (6).

Application4: If a differential equation is like

(7) $$dy_t = -\frac{1}{2} y_t dt + \sqrt{1 - y_t^2} dB_t, \; t < \inf \left\{ s > 0, \; B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}$$

Then the solution is

(7') $$y_t = \sin B_t, \; B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

The demonstration is similar; we apply Itô's formula to the function $f(t,x) = \sin x$

Application 5: The solution for the stochastically differential system

(8) $$\left[\begin{array}{c} dy_1 \\ dy_2 \end{array} \right] = \left[\begin{array}{c} 1 \\ y_2 \end{array} \right] dt + \left[\begin{array}{c} 0 \\ e^{y_1} \end{array} \right] dB_t$$

(8')
$$is\ y_t = (y_1, y_2)^t = \left(t, e^t B_t\right)^t.$$

Demonstration: By applying Itô's formula for the stochastically differential calculation of the process $(y_t)_{t\in R}$ where $y_t = (y_1, y_2)^t = (f_1(t,x), f_2(t,x))^t$, and $f_1(t,x) = t,\ f(t,x) = xe^t,\ x = B_t$

Where $(B_t)_{t\in R}$ is a Brownian 1dimensional Brownian movement

Application 6:Let be n-dimensional Brownian movement $B_t = (B_1(t), ..., B_n(t))^T$ of R^n and $\alpha_1, ..., \alpha_n$, real constants. Then the solution for the stochastically differential

(9)
$$dy_t = ry_t dt + y_t \left(\sum_{k=1}^{n} \alpha_k dB_k(t)\right),\ \ y_0 > 0$$

(9')
$$is\ y_t = e^{\sum\limits_{k=1}^{n} \alpha_k B_k(t)}$$

Demonstration: We apply Itô's formula for stochastically differential of the process $(y_t)_{t\in R}$ where

$$y_t = f(t, B_1(t), ..., B_n(t)), f(t, x_1, ...x_n) = e^{\alpha_1 x_1 + ... + \alpha_n x_n},$$

so that

$$dy_t = f_t' dt + \sum_{k=1}^{n} f_{x_k}' dx_k + \frac{1}{2}\sum_{i,j}^{n} f_{x_l x_j}'' dx_i dx_j$$

$$dy_t = 0dt + \sum_{k=1}^{n} \alpha_k e^{\alpha_1 x_1 + ... + \alpha_n x_n} dx_k + \frac{1}{2}\sum_{i,j}^{n} \alpha_i \alpha_j e^{\alpha_i x_1 + ... + \alpha_n x_n} dx_i dx_j$$

We note $\sum\limits_{i=1}^{n} \alpha_i^2 = r$ and we obtain (9')

Conclusions: Stochastically equation are solved are just some exemplas from the large array of applicability of Itô's formula, that represents a probabilistically abordation of high accuracy, into a realistic vision of dynamical evaluative systems. The presented cases are degree 1 equations and the found solutions are periodical, rational and exponential of Brownian movement functions.

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