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LINEAR PERTURBATIONS FOR THE DIRICHLET PROBLEM

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Abstract. In this paper we study a problem of type

$$\begin{cases} -\Delta u + u + Au = f & in \quad D \\ u = 0 & on \quad \partial D \end{cases}$$

which generalizes a result from [3], p.175. The results are given in theorem 1.1 and for case with bounded domain of class C^1, in theorem 2.2. Remark that these results remains true in case A nonlinear and the above problem can be establish also for variational inequalities ([3], p.118-120). Nonlinear case is studied using the author's results from [5].

Let $D \subset R^N$ be open and let $A : D(A) \subset L^2(D) \to L^2(D)$ be an operator. We consider the following problem:

$$\begin{cases} -\Delta u + u + Au = f & in \quad D \\ u = 0 & on \quad \partial D \end{cases} . \tag{1.1}$$

Case $A = 0$ is studied for example in [3].

Definition 1.1. *$u \in H_0^1(D)$ is called weak solution of the problem (1) if*

$$\int_D \nabla u \nabla v + \int_D uv + \int_D Au \cdot v = \int_D fv \quad , \quad \forall v \in H_0^1(D). \tag{1.2}$$

In the sequel, we give an existence and uniqueness result of weak solution for problem 1.1, then we see the conditions to obtain classical solution.

Theorem 1.1. *Assume that $A : D(A) \subset L^2(D) \to L^2(D)$ is linear, continous and monotone. Then for every $f \in L^2(D)$, the problem 1.1 has an unique weak solution denoted $u \in H_0^1(D)$. Moreover, if A is selfadjoint, then u realizes*

$$\min_{v \in H_0^1(D)} \left\{ \frac{1}{2} \int_D \left(|\nabla v|^2 + v^2 + Av \cdot v \right) - \int_D fv \right\}.$$

We will use the following

Theorem 1.2 (Lax-Milgram). *Let H be a real Hilbert space and let $a : H \times H \to \mathbf{R}$ be a bilinear, continuous and coercive form. Then for every $f \in H'$ there exists an unique $u \in H$ such that*

$$a(u, v) = f(v) \quad , \quad \forall v \in H.$$

Moreover, if a is symmetric, then u realizes

$$\min_{v \in H} \left\{ \frac{1}{2} a(v, v) - f(v) \right\}. \tag{1.3}$$

For proof, it is used *Riesz* representation theorem and *Picard-Banach* theorem (see [3], p.84).

Proof of theorem 1.1. Let us consider the Hilbert space $H = H_0^1(D)$ endowed with the scalar product

$$(u,v)_{H_0^1} = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2}. \tag{1.4}$$

The induced norm is

$$\|u\|_{H_0^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 . \tag{1.5}$$

Define on H the bilinear form $a : H \times H \to \mathbf{R}$,

$$a(u,v) = \int_D \nabla u \nabla v + \int_D uv + \int_D Au \cdot v. \tag{1.6}$$

We prove that a is continuous and coercive. Indeed,

$$a(u,v) = (\nabla u, \nabla v)_{L^2} + (u,v)_{L^2} + (Au,v)_{L^2}, \tag{1.7}$$

or equivalent $a(u,v) = (u,v)_{H_0^1} + (Au,v)_{L^2}$. It results

$$|a(u,v)| \le |(u,v)|_{H_0^1} + |(Au,v)|_{L^2} \le \|u\|_{H_0^1} \|v\|_{H_0^1} + \|Au\|_{L^2} \|v\|_{L^2} \le$$
$$\le \|u\|_{H_0^1} \|v\|_{H_0^1} + c\,\|u\|_{L^2} \|v\|_{L^2} \le \|u\|_{H_0^1} \|v\|_{H_0^1} + c\,\|u\|_{H_0^1} \|v\|_{H_0^1} \le$$
$$\le (c+1)\,\|u\|_{H_0^1} \|v\|_{H_0^1} .$$

So $a(u,v)$ is continuous (we use that $\|u\|_{L^2} \le \|u\|_{H_0^1}$). Further,

$$a(v,v) = \|v\|_{H_0^1}^2 + (Av,v)_{L^2} \ge \|v\|_{H_0^1}^2 .,$$

thus a is coercive. Let us define $f \in H'$ by

$$f(v) = \int_D fv \quad , \quad \forall v \in H. \tag{1.8}$$

Now, we can apply *Lax-Milgram* theorem with the bilinear form $a(u,v)$ given by 1.6 and $f \in H'$ given by 1.8. Hence there exists an unique element $u \in H_0^1(D)$ satisfying

$$a(u,v) = f(v) \quad , \quad \forall v \in H_0^1(D), \tag{1.9}$$

which is equivalent with 1.2. Finally, u is weak solution of 1.1. If moreover, A is selfadjoint, then a is symmetric and consequently, relation 1.3 holds . $\square$

Under some conditions, the weak solution of the problem 1.1 is in fact classical solution, as we can see from the following

Theorem 1.3. *Assume that $D \subset \mathbf{R}^N$ is of class C^1 and let $f \in C(\overline{D})$. Let $u \in H_0^1(D)$ be a weak solution of the problem 1.1. If moreover, $u \in C^2(\overline{D})$, then u is classical solution of* 1.1.

Proof. From the fact that $u \in H_0^1(D) \cap C(\overline{D})$, it results that $u = 0$ on ∂D ([3], p.171). But u is weak solution of 1.1 and $H_0^1(D)$ is dense in $L^2(D)$, thus

$$\int_D (-\Delta u + u + Au - f)v = 0 \quad , \quad \forall v \in H_0^1(D). \tag{1.10}$$

It folows: $-\Delta u + u + Au - f = 0$ *a.e. in* D. From continuity, it obtains:

$$-\Delta u + u + Au = f \quad in \quad D, \tag{1.11}$$

so u is classical solution of the problem 1.1. □

If change in A with $A - I$ in 1.1, then we obtain another form of the problem 1.1:

$$\begin{cases} -\Delta u + Au = f \quad in \quad D \\ \quad u = 0 \quad on \quad \partial D \end{cases}. \tag{1.12}$$

We give the following:

Theorem 1.4. *Let A be linear, continuous such that $A - I$ is monotone. Then for every $f \in L^2(D)$, problem 1.12 has an unique solution denoted $u \in H_0^1(D)$. Moreover, if A is selfadjoint, then u realizes*

$$\min_{v \in H_0^1(D)} \left\{ \frac{1}{2} \int_D (|\nabla v|^2 + v \cdot Av) - \int_D fv \right\}. \tag{1.13}$$

The particular case $A = I$ in theorem 1.4 is studied in [3], p.175.

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