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A MULTIFACTORIAL MODEL

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Abstract. The multifactorial models refer to the dependence of a bond on several parameters, unlike the unifactorial ones depending on the interest rate only. We suppose that the value P of a bond is dependent on two random factors: the interest rate r and the volatility σ. Our purpose is to find a procedure to offer the bond value at a specific moment in time. A numerical method is indicated in this respect.

Key words: bond, interest rate, volatility, stochastic differential equation, Itô formula.

1. Introduction

A bond is a long term contract between two persons. The emitent has the obligation to pay to the holder at the maturity time a nominal value and the holder pays at the moment when the contract is activated a certain premium.

The bond is a financial instrument used to enhance the capital and its life duration is about ten years or more. On a short range the rate can be consider deterministic, but in the long run, it has a stochastic evolution.

The differential equation whose solution is the bond value P is:

$$\frac{\partial P}{\partial t} + \frac{1}{2}b^2 \cdot \frac{\partial^2 P}{\partial r^2} + (a - \lambda b)\frac{\partial P}{\partial r} - rP = 0, \quad t < T,$$

where $P(T;T) = 1$ is the final condition, λ is the risk price supposed constant, and the stochastic dynamic of the interest rate is:

$$\mathrm{d}r = a \ \mathrm{d}t + b\mathrm{d}B,$$

where a, b are the instant expectation and the instant variation, and B is a standard brownian motion; r is the interest rate and T denotes the maturity time.

2. A Multifactorial Model

The stochastic differential equations are:

$$\begin{cases} d\sigma = \alpha \, dt + \beta dB, \\ dr = \gamma \, dt + \sigma dB, \end{cases}$$

where r is the interest rate, σ is the stochastic volatility, α, β, γ are constants and B is the above - mentioned brownian motion; t denotes the actual moment of time, t¡T, with T the maturity moment.

Use the bidimensional Itô formula, where $P = P(t, r, \quad \sigma)$ it follows that:

$$dP = \left(\frac{\partial P}{\partial t} + \alpha\frac{\partial P}{\partial \sigma} + \gamma\frac{\partial P}{\partial r} + \frac{1}{2}\beta^2 \cdot \frac{\partial^2 P}{\partial \sigma^2} + \frac{1}{2}\sigma^2 \cdot \frac{\partial^2 P}{\partial r^2} + \beta\sigma\frac{\partial^2 P}{\partial \sigma \, \partial r} \right) dt$$

$$+ \left(\beta\frac{\partial P}{\partial \sigma} + \sigma \ \frac{\partial P}{\partial r} \right) dB.$$

The coefficient of $\mathrm{d}t$ is denoted by $\mu_P P$ and that of $\mathrm{d}B$ is $\sigma_P P$, where μ_P and σ_P are the expectation P respectively the variation of the stochastic process P.

Since:

$$\lambda = \frac{\mu_P - r}{\sigma_P},$$

with λ denoting the risk price, the following equation holds:

$$\frac{\partial P}{\partial t} + (\alpha - \lambda\beta)\frac{\partial P}{\partial \sigma} + (\gamma - \lambda\sigma)\frac{\partial P}{\partial r} + \frac{1}{2}\beta^2 \cdot \frac{\partial^2 P}{\partial \sigma^2} + \frac{1}{2}\sigma^2 \cdot \frac{\partial^2 P}{\partial r^2} + \beta\sigma\frac{\partial^2 P}{\partial\sigma\,\partial r} - rP = 0,$$

where $\sigma,\ r \in (0,\ 1),\ t \in [0,\ T],\ \alpha,\ \beta,\ \gamma,\ \lambda$ - constants, $\alpha,\ \beta,\ \gamma,\ \lambda \neq 0$, with the final condition $P(T, r,\ \sigma) = 1$.

This equation will be solved numerically using the ***explicit scheme algorithm*** in the next section.

3. The Finite Difference Method. The Explicit Scheme

For a certain real function $u = u(x, y, t)$ let's consider the equation:

$$\frac{\partial u}{\partial t} = A \cdot \frac{\partial^2 u}{\partial x^2} + 2B \cdot \frac{\partial^2 u}{\partial x \partial y} + C \cdot \frac{\partial^2 u}{\partial y^2}.$$

In order to associate this equation to a finite difference scheme, the existence domain of the solution is covered by a rectangular network of lines parallel to the coordinate axes, with the paces h_1 and h_2 on the spatial axes and p the pace for the temporal coordinates.

The points of the network (X, Y, T), are given by:

$X = jh_1,\ Y = kh_2,\ T = lp,\ j, k, l \in \mathbf{Z}_+$.

With the following notations:

$$U^l_{j,k} = U(j\,h_1,\ k\,h_2, lp),$$

$$\Phi^l_{j,k} = \frac{A}{h_1^2}\,(U^l_{j+1,k} - 2U^l_{j,k} + U^l_{j-1,k})$$

$$+\frac{B}{2h_1h_2}\,(U^l_{j+1,k+1} - U^l_{j-1,k+1} - U^l_{j+1,k-1} + U^l_{j-1,k-1})$$

$$+\frac{C}{h_2^2}\,(U^l_{j,k+1} - 2U^l_{j,k} + U^l_{j,k-1}).$$

The finite difference scheme is:

$$\frac{U^{l+1}_{j,k} - U^l_{j,k}}{p} = \theta \cdot \Phi^{l+1}_{j,k} + (1-\theta) \cdot \Phi^l_{j,k}, \quad j = \overline{0, J},\ \ k = \overline{0, K},\ \ l = \overline{0, J},\ \ \theta \in [0, 1].$$

For $\theta = 0$ we have an explicit scheme that is stable for p very small.

The following algorithm in MATHCAD offers a numerical solution for the equation at the end of section 2, using the explicit scheme:

$\alpha := 0.5 \quad \beta := 0.5 \quad \gamma := 0.5 \quad \lambda := 0.5$

$$f(r, s, u, ur, us, u2r, urs, u2s) := r \cdot u + (\lambda \cdot s - \gamma) \cdot ur + (\lambda \cdot \beta - \alpha) \cdot us -$$

$$-0.5 \cdot \beta^2 \cdot u2s - \beta \cdot s \cdot urs - 0.5s^2 \cdot u2r$$

n := 10 r0 := 0 rf := 1 s0 := 0 sf := 1 T := 1 M := 10

$$hr := \frac{rf - r0}{n} \quad hr := \frac{sf - s0}{n} \quad \tau := \frac{T}{M}$$

$$
\begin{array}{l}
sol := \left| \begin{array}{l}
dim \leftarrow 2 \cdot M + n + 1 \\
for\ i \in 0..2 \cdot M + n \\
\quad for\ j \in 0..2 \cdot M + n \\
\quad\quad w_{i,j} \leftarrow 1 \\
\quad for\ k \in 1..M \\
\quad \left| \begin{array}{l}
t \leftarrow T - (k-1) \cdot \tau \\
v \leftarrow submatrix[w, 0, dim - 1, (k-1) \cdot dim, k \cdot dim - 1] \\
for\ i \in 0..2 \cdot M + n - 2 \cdot k \\
\quad \left| \begin{array}{l}
r \leftarrow r0 + (i - M + k) \cdot hr \\
for\ j \in 0..2 \cdot M + n - 2 \cdot k \\
\quad \left| \begin{array}{l}
s \leftarrow s0 + (j - M + k) \cdot hs \\
u \leftarrow v_{i+1,j+1} \\
ur \leftarrow \frac{v_{i+2,\ j+1} - v_{i,\ j+1}}{2 \cdot hr} \\
us \leftarrow \frac{v_{i+1,\ j+2} - v_{i+1,\ j}}{2 \cdot hs} \\
u2r \leftarrow \frac{v_{i+2,\ j+1} - 2 \cdot v_{i+1,\ j+1} + v_{i,\ j+1}}{hr^2} \\
urs \leftarrow \frac{v_{i+2,\ j+2} + v_{i,\ j} - v_{i+2,\ j} - v_{i,\ j+2}}{4 \cdot hr \cdot hs} \\
u2s \leftarrow \frac{v_{i+1,\ j+2} - v_{i+1,\ j+1} + v_{i+1,\ j}}{hs^2} \\
z_{i,j} \leftarrow v_{i+1,j+1} - \tau \cdot f(r, s, u, ur, us, u2r, urs, u2s)
\end{array} \right.
\end{array} \right. \\
for\ i \in 2 \cdot M + n - 2 \cdot k + 1..2 \cdot M + n \\
\quad for\ j \in 2 \cdot M + n - 2 \cdot k + 1..2 \cdot M + n \\
\quad\quad z_{i,j} \leftarrow 0 \\
w \leftarrow argument(w, z)
\end{array} \right. \\
w
\end{array} \right. \\
nivel(k) := \left| \begin{array}{l}
dim \leftarrow 2 \cdot M + n + 1 \\
d \leftarrow 2 \cdot M + n - 2 \cdot k + 1 \\
submatrix[submatrix[sol, 0, dim - 1, k \cdot dim, (k+1) \cdot dim - 1], 0, d - 1, 0, d - 1]
\end{array} \right.
\end{array}
$$

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