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THE CONSTRUCTION OF A STANCU TYPE OPERATOR

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ABSTRACT. Using the Taylor series expansion around 0 of the Stancu operator we construct a new linear and positive operator $R_m : C([0,1]) \to C([0,1])$ and we give a convergence theorem.

The Bernstein approximations $B_m f$, $m \in \mathbb{N}$ associated to a given continuous function $f : [0,1] \to \mathbb{R}$ is the polynomial

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{2}$$

are called the Bernstein fundamental polynomials of $m-$th degree (see [1]).

In 1968, D.D. Stancu defined in [5] a linear positive operator depending on two non-negative parameters α and β satisfying the condition $0 \le \alpha \le \beta$. Those operators defined for any non-negative integer m, associate to every function $f \in C([0,1])$ the polynomial $P_m^{(\alpha,\beta)} f$,

$$f \in C([0,1]) \longmapsto P_m^{(\alpha,\beta)} f,$$

in the following way:

$$(P_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right). \tag{3}$$

Note that for $\alpha = \beta = 0$ the Bernstein-Stancu operators become the classical Bernstein operators B_m. It is known that the Bernstein-Stancu operators verify the following relations:

Lemma 1 *For Bernstein-Stancu operators* $P_m^{(\alpha,\beta)}$, $m \in \mathbb{N}$, *the following relations hold true:*

1) $(P_m e_0)(x) = 1$

2) $(P_m e_1)(x) = x + \dfrac{\alpha - \beta x}{m+\beta}$

3) $(P_m e_2)(x) = x^2 + \dfrac{mx(1-x) + (\alpha - \beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2}$,

where

$$e_j(x) = x^j \quad , \quad j = 0, 1, 2$$

are test functions.

Lemma 2 *For any $x \in [0,1]$ we have:*
1.

$$(P'_m f)(x) = \frac{m}{m+\beta} \sum_{k=0}^{m-1} p_{m-1,k}(x)[a_k, a_{k+1}; f] \tag{4}$$

where $a_k = \frac{k+\alpha}{m+\beta}$, $k = \overline{0,m}$.
2.

$$\frac{1}{j!}\frac{d^j}{dx^j}(P_m f)(x) = \binom{m}{j} \frac{j!}{(m+\beta)^j} \sum_{k=0}^{m-j} p_{m-j,k}(x)[a_k, a_{k+1}, \ldots, a_{k+j}; f], \tag{5}$$

for $j \le m$.

Proof:

1. We use the relation:

$$p'_{m,k}(x) = m(p_{m-1,k-1}(x) - p_{m-1,k}(x)), \qquad k = \overline{0,m}, \qquad (\forall) x \in [0,1] \tag{6}$$

$$p_{0,0}(x) = 1, \qquad p_{s,-1}(x) = p_{s,s+1}(x) = 0, \qquad s \in N^* \tag{7}$$

so we can write:

$$(P'_m f)(x) = \sum_{k=0}^{m} p'_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) =$$

$$= m\sum_{k=0}^{m} p_{m-1,k-1}(x) f\left(\frac{k+\alpha}{m+\beta}\right) - m\sum_{k=0}^{m} p_{m-1,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right).$$

The first term of the first sum is zero, and so is the last term of the second sum (because of 7), therefore:

$$(P'_m f)(x) = m\sum_{k=1}^{m} p_{m-1,k-1}(x) f\left(\frac{k+\alpha}{m+\beta}\right) - m\sum_{k=0}^{m-1} p_{m-1,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) =$$

$$= m\sum_{l=0}^{m-1} p_{m-1,l}(x) f\left(\frac{l+1+\alpha}{m+\beta}\right) - m\sum_{k=0}^{m-1} p_{m-1,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) =$$

$$= m\sum_{l=0}^{m-1} p_{m-1,l}(x) \left(f\left(\frac{l+1+\alpha}{m+\beta}\right) - f\left(\frac{l+\alpha}{m+\beta}\right)\right) =$$

$$= m\sum_{l=0}^{m-1} p_{m-1,l}(x) \left(\frac{f\left(\frac{l+1+\alpha}{m+\beta}\right) - f\left(\frac{l+\alpha}{m+\beta}\right)}{\left(\frac{l+1+\alpha}{m+\beta} - \frac{l+\alpha}{m+\beta}\right)(m+\beta)}\right) =$$

$$= \frac{m}{m+\beta}\sum_{k=0}^{m-1} p_{m-1,k}(x) \left[\frac{k+\alpha}{m+\beta}, \frac{k+1+\alpha}{m+\beta}; f\right]$$

2. We use induction method. For $j = 1$ the above formula is obtained. Let's suppose that the formula 5 holds for $j-1$ and prove it for j.

$$\frac{1}{j!}\left(P_m^{(j)} f\right)(x) = \frac{1}{j}\left(\frac{1}{(j-1)!} P_m^{(j-1)} f\right)'(x) =$$

$$= \frac{1}{j}\binom{m}{j-1} \frac{(j-1)!}{(m+\beta)^{j-1}} \sum_{k=0}^{m-j+1} p'_{m-j+1,k}(x)[a_k, a_{k+1}, \ldots, a_{k+j-1}; f] =$$

$$= \frac{1}{j}\binom{m}{j-1}\frac{(j-1)!}{(m+\beta)^{j-1}}(m-j+1)\cdot$$

$$\cdot\left(\sum_{k=0}^{m-j+1} p_{m-j,k-1}(x)[a_k,a_{k+1},\ldots,a_{k+j-1};f] - \sum_{k=0}^{m-j+1} p_{m-j,k}(x)[a_k,a_{k+1},\ldots,a_{k+j-1};f]\right)$$

The first term of the first sum is zero, and so is the last term of the second sum, so we have:

$$\frac{1}{j!}\left(P_m^{(j)}f\right)(x) =$$

$$= \binom{m}{j}\frac{(j-1)!}{(m+\beta)^{j-1}}\left(\sum_{k=1}^{m-j+1} p_{m-j,k-1}(x)[a_k,a_{k+1},\ldots,a_{k+j-1};f] - \right.$$

$$\left. - \sum_{k=0}^{m-j} p_{m-j,k}(x)[a_k,a_{k+1},\ldots,a_{k+j-1};f]\right) =$$

$$= \binom{m}{j}\frac{(j-1)!}{(m+\beta)^{j-1}}\left(\sum_{l=0}^{m-j} p_{m-j,l}(x)[a_{l+1},a_{k+2},\ldots,a_{l+j};f] - \right.$$

$$\left. - \sum_{k=0}^{m-j} p_{m-j,k}(x)[a_k,a_{k+1},\ldots,a_{k+j-1};f]\right) =$$

$$= \binom{m}{j}\frac{(j-1)!}{(m+\beta)^{j-1}}\sum_{l=0}^{m-j} p_{m-j,l}(x)\left([a_{l+1},a_{k+2},\ldots,a_{l+j};f] - [a_l,a_{l+1},\ldots,a_{l+j-1};f]\right) =$$

$$= \binom{m}{j}\frac{(j-1)!}{(m+\beta)^{j-1}}\sum_{l=0}^{m-j} p_{m-j,l}(x)\left(\frac{[a_{l+1},a_{k+2},\ldots,a_{l+j};f] - [a_l,a_{l+1},\ldots,a_{l+j-1};f]}{a_{l+j}-a_l}\right)\cdot$$

$$\cdot(a_{l+j}-a_l) = \binom{m}{j}\frac{j!}{(m+\beta)^j}\sum_{k=0}^{m-j} p_{m-j,k}(x)[a_k,a_{k+1},\ldots,a_{k+j};f].$$

Lemma 3 *The following relation holds:*

$$(P_m f)(x) = \sum_{j=0}^{m}\binom{m}{j}\frac{j!}{(m+\beta)^j}x^j[a_0,a_1,\ldots,a_j;f]. \tag{8}$$

Proof We develop the polynomials $P_m f$ in Taylor series around $x = 0$:

$$(P_m f)(x) = \sum_{j=0}^{m}\frac{\left(P_m^{(j)}f\right)(0)}{j!}x^j =$$

$$= \sum_{j=0}^{m}\binom{m}{j}\frac{j!}{(m+\beta)^j}x^j\sum_{k=0}^{m-j} p_{m-j,k}(0)[a_k,a_{k+1},\ldots,a_{k+j};f]$$

and since $p_{m-j,k}(0) = 0$ for all $k = \overline{0, m-j}$, and $p_{m-j,0}(0) = 1$ only the first term of the sum remains:

$$(P_m f)(x) = \sum_{j=0}^{m}\binom{m}{j}\frac{j!}{(m+\beta)^j}x^j[a_0,a_1,\ldots,a_j;f].$$

We construct the following operator $R_m : C([0,1]) \to C([0,1])$ given by:

(9) $$\left(R_m f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{k!}{(m+\beta)^k} t_{k,m}[a_0, a_1, \ldots, a_k; f] x^k$$

where the numbers $t_{k,m}$, $k = \overline{0,m}$ will be determined by the conditions of liniarity and positivity of operators R_m.

Theorem 4 *If* $e_j(x) = x^j$, $j = 0, 1, 2$ *are the test functions then:*

1. $\left(R_m e_0\right)(x) = t_{0,m}$

2. $\left(R_m e_1\right)(x) = \frac{1}{m+\beta}\left(\alpha t_{0,m} + m t_{1,m} x\right)$

3. $\left(R_m e_2\right)(x) = \frac{1}{(m+\beta)^2}\left(\alpha^2 t_{0,m} + m(2\alpha+1) t_{1,m} x + m(m-1) t_{2,m} x^2\right)$

Proof 1. If $P \in \Pi_m$ is a polynomial of degree at most m, then for any x_0, $x_1, \ldots, x_{m+1}$ we have $[x_0, x_1, \ldots, x_{m+1}; P] = 0$, so the sum in the expression of $R_m e_0$ has only one term

$$\left(R_m e_0\right)(x) = t_{0,m}[a_0; e_0] = t_{0,m} e_0(a_0) = t_{0,m};$$

2. Using the same observation when computing $R_m e_1$ we obtain 2 terms:

$$\left(R_m e_1\right)(x) = t_{0,m}[a_0; e_1] + \frac{m}{m+\beta} t_{1,m}[a_0, a_1; e_1] x =$$

$$= t_{0,m} a_0 + \frac{m}{m+\beta} t_{1,m} \frac{e_1(a_1) - e_1(a_0)}{a_1 - a_0} x = \frac{1}{m+\beta}\left(\alpha t_{0,m} + m t_{1,m} x\right)$$

3.

$$\left(R_m e_2\right)(x) = t_{0,m}[a_0; e_2] + \frac{m}{m+\beta} t_{1,m}[a_0, a_1; e_2] x + \frac{m(m-1)}{(m+\beta)^2} t_{2,m}[a_0, a_1, a_2; e_2] x^2 =$$

$$= \frac{1}{(m+\beta)^2}\left(\alpha^2 t_{0,m} + m(2\alpha+1) t_{1,m} x + m(m-1) t_{2,m} x^2\right)$$

Lemma 5 *If* $\lim\limits_{m\to\infty} t_{0,m} = 1$ *and* $\lim\limits_{m\to\infty} t_{1,m} = 1$ *then* $\lim\limits_{m\to\infty} t_{2,m} = 1$.

Proof If $x \in [0,1]$ then $e_1(x) \geq e_2(x)$ and because R_m has to be a linear and positive operator it has to satisfy the inequality $\left(R_m e_1\right)(x) \geq \left(R_m e_2\right)(x)$, that is:

$$\frac{1}{m+\beta}\left(\alpha t_{0,m} + m t_{1,m} x\right) \geq \frac{1}{(m+\beta)^2}\left(\alpha^2 t_{0,m} + m(2\alpha+1) t_{1,m} x + m(m-1) t_{2,m} x^2\right).$$

For $m \to \infty$ we obtain:

(10) $$\lim_{m\to\infty} t_{1,m} \geq x \lim_{m\to\infty} t_{2,m}$$

that holds for any $x \in [0,1]$.

We consider $\varphi_x(t) = (t-x)^2$. Then $\left(R_m \varphi_x\right)(x) \geq 0$, and

$$\left(R_m \varphi_x\right)(x) = \left(R_m e_2\right)(x) - 2x\left(R_m e_1\right)(x) + x^2\left(R_m e_0\right)(x) =$$

$$= \frac{1}{(m+\beta)^2}\left(\alpha^2 t_{0,m} + m(2\alpha+1) t_{1,m} x + m(m-1) t_{2,m} x^2\right) -$$

$$-\frac{2x}{m+\beta}\left(\alpha t_{0,m} + m t_{1,m} x\right) + x^2 t_{0,m} \geq 0.$$

Making $m \to \infty$ it follows that

$$x^2 \lim_{m\to\infty}\left(t_{2,m} - 2t_{1,m} + t_{0,m}\right) \geq 0$$

or

$$\lim_{m\to\infty}\left(t_{2,m} - t_{1,m}\right) \geq \lim_{m\to\infty}\left(t_{1,m} - t_{0,m}\right) = 0$$

(11) $$\lim_{m\to\infty} t_{2,m} \geq \lim_{m\to\infty} t_{1,m}$$

From 10 and 11 we have that $\lim_{m\to\infty} t_{2,m} = \lim_{m\to\infty} t_{1,m} = 1$.

We will use the following result due to H. Bohman and P.P. Korovkin.

Theorem 6 *Let* $L_m : C([a,b]) \to C([a,b])$, $m \in \mathbb{N}$ *be a sequence of linear, positive operators such that*

$$(L_m e_0)(x) = 1 + u_m(x)$$
$$(L_m e_1)(x) = x + v_m(x)$$
$$(L_m e_2)(x) = x^2 + w_m(x)$$

with

$$\lim_{m\to\infty} u_m(x) = \lim_{m\to\infty} v_m(x) = \lim_{m\to\infty} w_m(x) = 0,$$

uniformly on $[0,1]$. *Then for every continuous function* $f \in C([0,1])$, *we have*

$$\lim_{m\to\infty} (L_m f)(x) = f(x),$$

uniformly on $[0,1]$.

We apply theorem 6 to the results obtained in theorem 4 and lemma 5 and we get the convergence theorem for the R_m operators given by relation 9.

Theorem 7 *If* $\lim_{m\to\infty} t_{0,m} = 1$ *and* $\lim_{m\to\infty} t_{1,m} = 1$ *then for every continuous function* $f \in C([0,1])$ *we have*

$$\lim_{m\to\infty} (R_m f)(x) = f(x),$$

uniformly on $[0,1]$.

References

[1] S.N. Bernstein, *Démonstration du théorème deWeierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow, 13(2) (1912-1913), 12.

[2] H. Bohman, *On approximation of continuous and analytic functions*, Ark. Mat., 2 (1952), 43-56.

[3] P.P. Korovkin, *On convergence of linear operators in the space of continuous functions* (Russian), Dokl. Akad. Nauk. SSSR (N.S.), 90 (1953), 961-964.

[4] G.G. Lorentz, *Bernstein Polynomials*, Toronto, Univ. of Toronto Press, 1953.

[5] D.D. Stancu, *Approximation of function by a new class of polynomial operators*, Rev. Roum. Math. Pures et Appl., 13(8) (1968), 1173-1194.

[6] D.D. Stancu, Gh. Coman, O. Agratini, R. Trâmbiţaş, *Analiză numerică şi teoria aproximării*, vol. I. Presa Universitară Clujeană, Cluj-Napoca, 2001 (in Romanian).

[7] D. D. Stancu, *Approximation properties of a class of linear positive operators*, Studia Univ. Babes-Bolyai Ser. Math.-Mech. 15 (fasc. 2) (1970) 33-38.

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