SOME GRONWALL TYPE INEQUALITIES WITH APPLICATION IN DATA DEPENDENCE FOR FUNCTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we study, with abstract Gronwall Lemma, the following inequality

$$x(t)e^{\omega t} \le A + B\int_0^t e^{\omega s}x(\theta s)ds,$$

where $A, B, \omega > 0$, $x \in C([0,T],(R)_+^*)$, $\theta \in (0,1)$. In the second part, we present the application of this for the following functional differential equation:

$$x'(t) - Ax(t) = B(t, x(\theta t))),$$

where

- $B \in C^1([0,+\infty) \times R^n)$.
- $A \in M_{n^2}(R)$ is a matrix with eigenvalues having negative real part.

1. INTRODUCTION

The class of the Picard operators was introduced by Profesor I.A. Rus.The notions and results are in I.A.Rus [6], [8], [7], [9].

Let (X,d) be a metric space and $A : X \to X$ an operator.We shall use the following notations:
$F_A := \{x \in X \mid Ax = x\}$ the fixed points set of A.
$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A.
$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in N$.

Definition 1.1. *(I.A.Rus [8]) An operator A is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n\in N}$$

converges , for all $x \in X$ and the limit (which depend on x) is a fixed point of A.

Definition 1.2. *(I.A.Rus [8]) If the operator A is WPO and $F_A = \{x^*\}$ then by definition A is Picard operator.*

Lemma 1.1. *(I.A. Rus [11])Let be $(X,d,\le)$ an ordered metrical space and $A, B : X \longrightarrow X$ two operators. We suppose that :*

(i) *A and B Picard operators;*
(ii) *A is increasing;*
iii *$A \le B$.*

Then $x \le A(x)$ imply $x \le x_B^*$.

2. Main Results

Proposition 2.1. *Fie $A, B, \omega > 0$, $\theta \in (0,1)$ and let $x \in C([0,T], \mathbb{R}_+)$ be a solution for the following inequality*

$$x(t)e^{\omega t} \leq A + B\int_0^t e^{\omega s}x(\theta s)ds.$$

Then

$$x(t) \leq Ae^{\frac{B}{\theta}t},\ (\forall) t \in [0,T].$$

Proof: On $C([0,T], \mathbb{R}_+)$, we define the following norm

$$\|x\|_\tau = \max_{t\in[0,T]} |x(t)|e^{-\tau t},\ \tau > 0$$

$$x(t) \leq Ae^{-\omega t} + B\int_0^t e^{-\omega(t-s)}x(\theta s)ds \leq A + B\int_0^t e^{-\omega(t-s)}x(\theta s)ds \leq$$

$$\leq A + B\int_0^t x(\theta s)ds = A + \frac{B}{\theta}\int_0^{\theta t} x(u)du \leq A + \frac{B}{\theta}\int_0^t x(u)du.$$

We define

$$M, N : C([0,T], \mathbb{R}_+) \longrightarrow C([0,T], \mathbb{R}_+)$$

$$M(x)(t) = Ae^{-\omega t} + B\int_0^t e^{-\omega(t-s)}x(\theta s)ds$$

$$N(x)(t) = A + \frac{B}{\theta}\int_0^t x(u)du.$$

We remark that $x \leq M(x) \leq N(x)$, M is increasing and for all $x, y \in C([0,T], \mathbb{R}_+)$ we have that

$$|M(x)(t) - M(y)(t)| \leq B\int_0^t |x(\theta s) - y(\theta s)|ds \leq$$

$$\leq B\|x - y\|_\tau \int_0^t e^{\tau\theta s}ds \leq \frac{B}{\tau}\|x - y\|_\tau e^{\tau t}.$$

Then

$$\|M(x) - M(y)\|_\tau \leq \frac{B}{\tau}\|x - y\|_\tau.$$

We choose $\tau > 0$ such that $\frac{B}{\tau} < 1$, and we obtain that M is the Picard operator. Analog we show that N is a Picard operator. The unique fixed point of operator N is $\overline{x}(t) = Ae^{\frac{B}{\tau}t}$. Q.E.D

Next we consider the following Cauchy problem

$$x'(t) - Ax(t) = B(t, x(\theta t)), t \in [0,T]. \tag{1}$$

$$x(0) = x_0. \tag{2}$$

where

(H_1) $B \in C^1([0,+\infty) \times R^n)$.

(H_2) $A \in M_{n^2}(R)$ is a matrix with eigenvalues having negative real part.

(H_3) There exists $L_B > 0$ such that

$$\|\frac{\partial B}{\partial u_i}(t,u)\|_{R^n} \leq L_B, (\forall) t \geq t_0, (\forall) u_i \in R.$$

The problem (1)+(2) is equivalent with

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} B(s, x(g(s)))ds, \tag{3}$$

Proposition 2.2. *We suppose that the hypothesis (H_1), (H_2), (H_3) are satisfied. Then*

(a) *The Cauchy problem (1)+(2) has a unique solution $x(\cdot, 0, x_0)$.*

(b) *If, there exists $\eta > 0$ such that $\|x_0 - y_0\|_{\mathbb{R}^n} \leq \eta$, then*

$$\|x(t,0,x_0) - x(t,0,y_0)\|_{\mathbb{R}^n} \leq M\eta e^{\frac{M}{\theta}T}$$

Proof:

Let $L : C([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ be defined as follows:

$$L(x)(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} B(s, x(g(s)))ds.$$

We have that

$$\|L(x) - L(y)\|_\tau \leq \frac{L_B}{\tau\theta}\|x - y\|_\tau$$

It follows that L is a Picard operator. From here we obtain (a).

We note by $h(t) = \|x(t,0,x_0) - x(t,0,y_0)\|_{\mathbb{R}^n} \in C([0,T], \mathbb{R}_+)$. Using the structure theorem of the matrix A we obtain that

$$h(t) \leq M\eta + M\int_0^t e^{\omega s} h(\theta s)ds.$$

From Proposition 2.2 we have that

$$h(t) \leq M\eta e^{\frac{M}{\theta}t} \leq M\eta e^{\frac{M}{\theta}T}$$

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