SOLVING THE SADDLE-POINT PROBLEM FOR THE QUASISTATIC CONTACT PROBLEMS

NICOLAE POP

ABSTRACT. The paper is concerned with the numerical solution of the quasi-variational inequality modelling a contact problem with Coulomb friction. After discretization of the problem by mixed finite elements and with Lagrangian formulation of the problem by choosing appropriate multipliers, the duality approach is improved by splitting the normal and tangential stresses. The novelty of our approach in the present paper consists in the splitting of the normal stress and tangential stress, which leads to a better convergence of the solution, due to a better conditioned stiffness matrix. This better conditioned matrix is based on the fact that the obtained diagonal blocks matrices, contain coefficients of the same size order. For the saddle point formulation of the problem, using static condensation, we obtain a quadratic programming problem.

Subject Classification: 35J85, 49J35, 65N30, 74M15, 74S05, 65Y20

Key words and phrase: Contact problem with Coulomb friction, dual mixed formulation, mixed finite element, saddle point problem, quadratic programming, Schur complement

1. CLASSICAL AND VARIATIONAL FORMULATION

In this paper we study a mathematical model of frictional quasistatic contact between a deformable body under consideration is assumed to be elastic with a linear elasticity operator and a foundation. The mathematical model consists in a hemivariational inequality which involves the Clarke subdifferential of a locally Lipschitz functional, (see [20]). The first step in the sequence of the approximations is the penalty method for to replace the unilateral contact conditions by a nonlinear boundary condition dependent on the small parameter. The second step is the regularization method for the approximation of the module function, with a convex function . Let $\Omega \subset \mathbb{R}^d, d = 2$ or 3, the domain occupied by a linear elastic body with a Lipschitz boundary Γ. Let Γ_1, Γ_2 and Γ_C be three open disjoint parts of Γ such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_C,\ \overline{\Gamma}_1 \cap \overline{\Gamma}_C = \varnothing$ and mes $(\Gamma_1) > 0$. We assume that the body is subjected to volume forces of density $\boldsymbol{f} \in (L^2(\Omega))^d$, to surface traction of density $\boldsymbol{h} \in (L^2(\Gamma_2))^d$ and is held fixed on Γ_1 . The Γ_C denotes a contact part of boundary where unilateral contact and Coulomb friction condition between Ω and perfectly rigid foundation are considered. We denote by $\boldsymbol{u} = (u_1, \ldots, u_d)$ the displacement field, $\boldsymbol{\varepsilon} = (\varepsilon_{ij}(\mathbf{u})) = \left(\frac{1}{2}(u_{i,j} + u_{j,i})\right)$ the strain tensor and $\boldsymbol{\sigma} = (\sigma_{ij}(\mathbf{u})) = (a_{ijkl}\varepsilon_{kl}(\mathbf{u}))$ the stress tensor with the usual summation convention, where $i, j, k, l = 1, \ldots, d$. For the normal and tangential components of the displacement vector and stress vector, we use the following notation: $\mathbf{u}_N = u_i \cdot n_i$, $\boldsymbol{u}_T = \boldsymbol{u} - \boldsymbol{u}_N \cdot \mathbf{n}$, $\boldsymbol{\sigma}_N = \boldsymbol{\sigma}_{ij} u_i n_j$, $(\boldsymbol{\sigma}_T)_i = \boldsymbol{\sigma}_{ij} n_j - \boldsymbol{\sigma}_N \cdot n_i$, where $\mathbf{n} = (n_i)$ is the outward unit normal vector to Γ.

We denote by $g \in C(\bar{\Gamma}_C)$, $g \geq 0$ the initial gap between the body and the rigid foundation and lets us denote by $\boldsymbol{f}$ and $\boldsymbol{h}$ the density of body and traction forces, respectively. We assume that $a_{ijkl} \in L^\infty(\Omega)$, $l \leq i, j, k, l \leq d$, with usual condition of symmetry and elasticity, that is

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad 1 \leq i, j, k, l \leq d,$$

$$\text{and} \quad \exists\, m_0 > 0,\ \forall\, \xi = (\xi_{ij}) \in \mathbb{R}^{d^2},\ \xi_{ij} = \xi_{ji},\ 1 \leq i,\ j \leq d,\ a_{ijkl}\, \xi_{ij}\, \xi_{kl} \geq m_0 |\xi|^2 .$$

In this conditions, the fourth-order tensor $\boldsymbol{a} = (a_{ijkl})$ is invertible a.e., on Ω and if we denote its inverse by $\boldsymbol{b} = (b_{ijkl})$, we have $\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})) = (b_{ijkl}\sigma_{kl}(\boldsymbol{u}))$, $i, j, k, l = 1, \ldots, d$.

The classical contact problem with dry friction in elasticity, in the particular case, is with the normal stress $\sigma_N(u)$ and Γ_C is assumed known and considered as obeying the normal compliance law, is the following

Find $\boldsymbol{u} = \boldsymbol{u}(x,t)$ such that $\boldsymbol{u}(0,\cdot) = \boldsymbol{u}^0(\cdot)$ in Ω and for all $t \in [0,T]$,

$$-\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega \tag{1.1}$$

$$\boldsymbol{\sigma}_{ij}(\boldsymbol{u}) = a_{ijkl} \cdot \varepsilon_{kl}(\boldsymbol{u}), \quad \text{in } \Omega \tag{1.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_1 \tag{1.3}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{h} \quad \text{on } \Gamma_2, \tag{1.4}$$

the contact condition:

$$u_N \leq g,\ \boldsymbol{\sigma}_N(u) \leq 0,\ (u_N - g)\boldsymbol{\sigma}_N(u) = 0 \quad \text{on } \Gamma_C \tag{1.5}$$

and Coulomb friction on Γ_C:

$$\begin{aligned} &\|\sigma_T(u)\| \leq \mu_F|\sigma_N(u)|, \text{ such that :}\\ &\quad - \text{if} \quad \|\sigma_T(u)\| < \mu_F|\sigma_N(u)| \Rightarrow u_T = 0\\ &\quad - \text{if} \quad \|\sigma_T(u)\| = \mu_F|\sigma_N(u)| \Rightarrow \exists\, \alpha \geq 0, \text{ such that } \dot{u}_T = -\alpha\sigma_T \end{aligned} \tag{1.6}$$

where $\boldsymbol{u}^0$ denotes the initial displacement of the body. Supposing that a positive coefficient $\mu_F \in L^\infty(\Gamma_C)$, $\mu_F \geq \mu_0$ a.e. on Γ_C of Coulomb friction is given, we introduce the space of *virtual displacements*

$$V = \left\{v \in (H^1(\Omega))^2 | v = 0 \text{ on } \Gamma_1\right\}$$

and its convex subset of *kinematically admissible displacements*

$$K = \{v_N \in V | v_N \equiv v \cdot n \leq g \text{ on } \Gamma_C\}.$$

We assume that the normal force on Γ_C is known (as normal compliance) so that one can evaluate the non-negative slip bound $p \in L^\infty(\Gamma_C)$ as a product of the friction coefficient and the normal stress, i.e. $p = \mu_F\lambda_1$, when λ_1 is the normal stress. We assume that normal interface response (the normal compliance law) is:

$$\sigma_N(u) = -c_N(u_N - g)^{m_N}$$

where c_N and m_N are material constant depending on interface properties.

(P_1) *Find* $u \in K$ *such that* $J(u) = \min_{v \in K} J(v)$.

The minimized functional representing the total potential energy of the body has the form:

$$J(v) = \frac{1}{2}a(v,v) - L(v) + \overline{j}(v)$$

where:

- the bilinear form a is given by

$$a(v,w) = \int_\Omega a_{ijkl}\varepsilon_{ij}(v)\varepsilon_{kl}(w)dx$$

- linear functional L is given by:

$$L(v) = \int_\Omega fvdx + \int_{\Gamma_2} hvds;$$

- the sublinear functional $\overline{j}$ is given by:

$$\overline{j}(v) = \int_{\Gamma_C} p\,|v_T|\,ds + \int_{\Gamma_C} c_N(u-g)^{m_n} v_N ds$$

where $v_T \in (L^\infty(\Gamma_C))^2$ denotes the tangent vector to boundary Γ.

It is known that the problem (P_1) is non-differentiable due to the sublinear term $\overline{j}$, and has a unique solution [9].

The variational formulation, in the quasi-static case, is equivalent to the quasi-variational inequality:

(P_2) *Find* $u(x,t) \in K \times [0,T]$ s. t. $a(u, v-\dot{u}) + \overline{j}(v-\dot{u}) \geq (L, u-\dot{v})\ \ \forall v \in K, \forall t \in [0,T], T>0$, with initial conditions $u(x,0) = u_0, \dot{u}(x,0) = u_1$.

The existence and uniqueness of the solution of this quasi-variational inequality are proven under the assumption that μ_F is sufficiently small and $mes(\Gamma_0) > 0$ [16].

The *Lagrangian formulation* of the problem (P_1) is given by introducing $L : V \times \Lambda_1 \times \Lambda_2 \to \mathbb{R}$, with

$$L(v, \mu_1, \mu_2) = \frac{1}{2}a(v,v) - L(v) + \langle \mu_1, v_N - g\rangle + \int_{\Gamma_C} \mu_2 v_T ds$$

where $\Lambda_1 = \{\mu_1 \in H^{-\frac{1}{2}}(\Gamma_C) | \mu_1 \geq 0\}$, $\Lambda_2 = \{\mu_2 \in L^\infty(\Gamma_C)|\ \ |\mu_2| \leq p \text{ on } \Gamma_C\}$.

The space $H^{-\frac{1}{2}}(\Gamma_C)$ is the dual of

$$H^{\frac{1}{2}}(\Gamma_C) = \{\gamma \in L^2(\Gamma_C) |\, \exists v \in V\ s.t.\ \gamma = v_N \text{ on } \Gamma_C\}$$

and the ordering $\mu_1 \geq 0$ means, in the variational form, that $\langle \mu_1, v_N - g\rangle \leq 0,\ \forall\, v \in K$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma_C)$ and $H^{\frac{1}{2}}(\Gamma_C)$. Since $L^2(\Gamma_C)$ is dense in $H^{-\frac{1}{2}}(\Gamma_C)$, the duality pairing $\langle \cdot, \cdot \rangle$ is represented by a scalar product in $L^2(\Gamma_C)$.

The Lagrange multipliers μ_1, μ_2 are considered as functionals on the contact part of the boundary Γ. It is important that the Lagrange multipliers do have mechanical significance: while the first one is related to the non-penetration conditions and represents the normal stress, the second one removes the non-differentiability of the sublinear functional

$$j_2(v) = \sup_{\mu_2 \in \Lambda_2} \int_{\Gamma_C} \mu_2 v_T ds$$

and represents the tangential stress.

The equivalence between the problem (P_1) and the lagrangian formulation is given by:

$$\inf_{v \in K} J(v) = \inf_{v \in V} \sup_{\mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2} L(v, \mu_1, \mu_2).$$

By the mixed variational formulation of the problem (P_1) we mean a saddle point problem:

$$\begin{array}{ll} (P_3) & \textit{find } (w, \lambda_1, \lambda_2) \in V \times \Lambda_1 \times \Lambda_2 \textit{ such that} \\ L(w, \mu_1, \mu_2) \leq L(w, \lambda_1, \lambda_2) \leq L(v, \lambda_1, \lambda_2), & \forall\, (v, \mu_1, \mu_2) \in V \times \Lambda_1 \times \Lambda_2. \end{array}$$

It is known that (P_3) has a unique solution [2] and its first component $w = u \in K$ solves (P_1) and the Lagrange multipliers λ_1, λ_2 represent the normal and tangential contact stress on the contact part of the boundary, respectively.

Remarks.

1^0. For the contact problem with Coulomb friction, we use the formula $p \equiv \mu_F \lambda_1$, for the slip bound on the contact boundary Γ_C , where $\lambda_1 \equiv\ \lambda_1(p)$ is the normal stress on Γ_C and μ_F is the coefficient of friction. Unfortunately this problem cannot be solved as a convex quadratic programming problem because p is an a priori parameter in (P_3), while λ_1 is an a posteriori one.

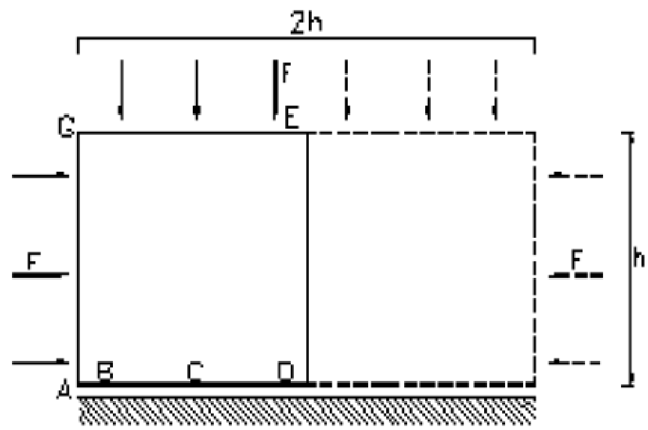

Fig 1. The geometry (h=40 mm) and the loading

μ	F daN/mm^2	f daN/mm^2	Separate part AB mm	Sliding part BC mm	Stick part CD mm
1	10	-5	3.75	20	16.25
1	15	-5	5	20.75	7.5
0.2	10	-5	0	40	0
0.2	10	-15	0	22.5	17.5
0.2	10	-25	0	5	35

Table 1. Contact states for different loading cases

References

[1] Anderson, L. E., *Approximation of the Signorini problem with friction obeying the Coulomb law*, Math. Methods Appl. Sci., **5** (1983), pp. 422-437

[2] Brezzi, F., Fortin., M., *Mixed and Hybrid Finite Element Method*, New York, Springer-Verlag, 1991

[3] Ciarlet, P. G., *The Finite Element Method for Elliptic Problems*, Amsterdam, New York, Oxford, North Holland, 1978

[4] Glowinski, R., Lions, J., Tremolieres, R., *Numerical Analysis of Variational Inequalities*, Amsterdam, New York, Oxford, North Holland, 1981

[5] Haslinger, J., *Approximation of the Signorini problem with friction obeying the Coulomb law*, Math. Methods Appl. Sci. **5** (1983), pp. 422-437

[6] Ju J.W., Taylor R.L., *A Perturbed Lagrangian Formulation for the Finite Element Solution of Nonlinear Frictional Contact Problem*, Journal de Mécanique Theoretique et Appliquée, Spec. Issue, suppl. to vol. **7** (1998), 1-14

[7] Kiruchi, N., Oden, J. T., *Contact problem in Elasticity: A Study of Variational Inequalities and Finite Element Method*, SIAM Philadelphia, 1988

[8] Klarbring, A., Mikelic, A., Shillor, M., *Global existence result for the quasistatic frictional contact problem with normal compliance*, In "Unilateral Problems in Structural Analysis IV (Capri 1989), 85-111, Birkhäuser, 1991

[9] Nečas, J., Jarušek, J., Haslinger, J., *On the solution of the variational inequality to the Signorini problem with small friction*, Boll. Un. Mat. Ital., B(6), **17** (1980), pp. 796-881

[10] Petrila, T., Trif, D., *Basic of Fluid Dynamics and Introduction to Computational Fluid Dynamics*, Springer, USA, 2005.

[11] Petrila, T., Ghiorghiu, C., Finite Element Methods and Applications (in Romanian), Ed. Acad. Rom., Bucuresti, 1987.

[12] Pop, N., *A Finite Element Solution for a Three-dimensional Quasistatic Frictional Contact Problem*, Rev. Roumaine des Sciences Tech. Serie Mec. Appliq. Editions de l'Academie Roumanie, Tom 42, 1-2, (1997), 209-218

[13] Pop, N., *On inexact Uzawa methods for saddle point problems arising from contact problem*, Bul. Ştiinţ. Univ. Baia Mare, Ser. B, Matematică-Informatică, **XV**, Nr. 1-2 (1999), pp. 45-54

[14] Raous M., Chabrand P., Lebon F., *Numerical methods for frictional contact problems and applications*, Journal de Mécanique Theoretique et Appliquée, Spec. issue, suppl. to vol. **7** (1998), 111-128

[15] Rocca, R., Cocou, M., *Numerical analysis of quasi-static unilateral contact problems with local friction*, SIAM J. NUMER. ANAL., **39**, No. 4, pp. 1324-1342, 2001

[16] Rocca, R., Cocou, M., *Existence and approximation of a solution to quasi-static Signorini problem with local friction*, Internat J. Eng. Sci. **39** (2001), pp. 1253-1258

[17] Wang, G., Wang, L., *Uzawa type algorithm based on dual mixed variational formulation*, Applied Mathematics and Mechanics, **23**, No.7, (2002), pp. 765-772

[18] Wohlmuth, I. B., Krause, H. R., *A multigrid method based on the unconstrained product space for mortar finite element discretetizations*, SIAM J. NUMER. ANAL., **39** (2001), No. 1, pp. 192-213

[19] Wriggers P., Simo J.C., *A note on tangent stiffness for fully nonlinear contact problems*, Comm. in App. Num. Math., **1** (1985), 199-203

[20] Bartosz K. *Hemivariational inequality approach to the dynamic viscoelastic sliding contact problem with wear*, Nonlinear Analysis, **65** (2006), pp.546-566

Department of Mathematics and Computer Science,
North University of Baia Mare,
Victoriei, 76, 430121 Baia Mare, Romania
E-mail address: nicpop@ubm.ro