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CONTINUATION METHOD AND TOPOLOGICAL INVARIATION

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Abstract. The inversability of an linear bounded operator can be deduced from the inversability of an known linear operator with help of continuation method, with important generation in the degree theory. In this report are presented continuation method, equations with sum A-proper, where topological degrees generalized can be used and the study of similar equations, of tip $Lx = S(x) + p$, in which $L$is Fredholm unbounded operator with zero index and $N(L) \neq \{O\}$, having the majority properties of coincidents degrees.

In the last part are given two examples: one in the case of approximation method for A-proper and another example with Hammerstein abstract equation.

1. Preliminaries

Continuation method:

• Let Xbe a Banach space, $y$an normate space and $L_0, L_1 \in L(X, Y)$ an bounded operators. We define the homotopi

L_t: = (1-t)$L_0 + tL_t$

and we suppose that there exist a constant $C > 0$, such that $\|x\|_X \leq C\|L_t x\|_y, \forall x \in X, t \in [0,1]$.

The equation with sum A-proper

Let be the equation of form

$$Lx = S(x) + p. \tag{1}$$

where $L : X \mapsto Y$ is a linear operator A-proper, so $S : \bar{D} \subseteq X \mapsto Y$an nonlinear operator with the property that the sum $T := L - S$ to be weak A-proper; $X$and Yseparable Banach spaces and $\Gamma_n := (\{x_n\}, \{y_n\}, \{P_n\}, \{Q_n\})$ the schem (of approximation) projective complet given.

Lemma: *If L is injective continuous operator, then the equation (1) has at last one solution for each $p \in Y$.*

We suppose that L is an Fredholm operator with $N(L) \neq \{0\}$ and index zero. In this case $\dim N(L) = co\dim\ R(L)$ and there exist closed subspaces $X_1 \subset X$ and $Y_1 \subset Y$ with $\dim Y_1 = \dim N(L)$, sach that $X = N(L) \oplus X_1$, $Y = Y_1 \oplus R(L)$, $L$is an injection on X_1, and $L(X_1) = R(L)$. Let M an izomorfism of $N(L)$ on Y_1, P the projection of Xon $N(L)$ and $C := MP$. Because C is a compact operator and L is a A-proper operator, it fallows like the operator $L_\lambda := L + \lambda C$ is, also A-proper. Mach more, L_λ is Fredholm closed operator with index zero.

Theorem. *We suppose that the image $T(\bar{B})$ is closed, any ball $B \subset X$and one of next sets,*

$$S^+ := \{x_n \in X;\ \ L_{\lambda_n} x_n = S(x_n) + p,\ \ \lambda_n \to 0_+\}$$

$$S^- := \{x_n \in X;\ \ L_{\lambda_n} x_n = S(x_n) + p,\ \ \lambda_n \to 0_-\}$$

is bounded by a constant wich doesn't depends of $n \geq N$, then the equation (1) has at last one solution for $p \in Y$.

Corollary. *Let X a reflexive space, K : X→ Y*an bounded operator, weak continuous, with R(K) dense in Y*such that* $Q_n^* K\,x = K\,x$, *L :X→Y an Fredholm operator of index zero, S : X →Y is asimptotic zero operator, wich satisface one of the next conditiones:*

(i) S is compact ;

(ii) S is weak continuous ;

(iii) S is semibounded.

If one of sets S^+, S^- *is bounded, then equation (1) has at last one solution for p∈ Y.*

We suppose now that $D_L := D \cap D(L) \neq \theta$ and the nonlinear operator $T := L - S : \bar{D}_L \mapsto Y$ is A-proper in raport with schem Γ_L made before. If $p \notin T(\partial D_L)$, then generalized degree $D_L(T,\ D,\ p) := D(T,\ D_L,\ p)$ is well defined and has the property from the precedent section.

Proposition (Continuons Theorems)

Let $S : [0,1] \times \bar{D} \mapsto Y$ *such that the application L-S* : $[0,1] \times \bar{D}_L \mapsto Y$ *to be a homotoic A-proper with respect to schem* Γ_L, *with S(l,.) = S. If Lx ≠ S(t,x)- t p,* $\forall t \in [0,1), x \in \partial D_L$ *and* $D_L(L - S(0,.), D, p) \neq \{0\}$, *then the equation (1) has at last one solution in* $\bar{D}_L$.

2. Examples

1. The continuation method for schem A-proper.

Let Ω a bounded domain in R^m. We consider Dirichelt problem for a elliptic equation of divergential type

$$(1)\ \begin{cases} -\sum\limits_{|\alpha|\leq 1} D^\alpha A_\alpha(x,u,u') + C(x,u,u') = f(x), & x \in \Omega \\ u\ (x) = 0 & x \in \partial\Omega \end{cases}$$

and we look to establish the existens of the generalized solution.

We note $\xi := \{\xi_\alpha \in R;\ |\alpha| \leq 1\} \in R^{1+m}$, $\xi(u) := \{D^\alpha u;\ |\alpha| \leq 1\}$

And we suppose that are fulfillment the next conditions:

(A_1) The functions $A_\alpha(x,\xi)$ satisfies Caratheodory conditions – they are measurable in $x \in \bar{\Omega}$ for each $\xi \in R^{1+m}$ and continue in ξ a.p.t. $x \in \bar{\Omega}$- and we admit grow up of type $|A_\alpha(x,\xi)| \leq k_0 |\xi|^{p-1} + h_0(x)$, $\forall x \in \bar{\Omega}$, $\xi \in R^{1+m}$, $|\alpha| \leq 1$, where $k_0 > 0$, $1 < p < \infty$, then $h_0 \in L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$;

(A_2) There exist the constants $c_0 \geq 0$ and $c_1 > 0$ such that

$$\sum_{|\alpha|\leq 1} \left[A_\alpha(x,\xi) - A_\alpha(x,\xi')\right](\xi - \xi') \geq c_1 \left(\sum_{|\alpha|\leq 1} |\xi_\alpha - \xi'_\alpha|^p\right) - c_0 |\xi_0 - \xi'_0|^p$$

any $x \in \bar{\Omega}$ and $\xi \neq \xi'$;

(A_3) For each $x \in \bar{\Omega}$, the function $A_\alpha(x,\ .) :\ R^m \mapsto R$ are old, $A_\alpha(x,\ -\xi) = -A_\alpha(x,\ \xi)$, and positive omogeneus order $s > 0$, $A_\alpha(x,\ t\xi) = t^s A_\alpha(x,\ \xi)$, $\forall t > 0$;

(C_1) The function $C(x,\xi)$ satisfaid Caratheodory condition and admits grow up $|C(x,\xi)| \leq k_1 |\xi|^{p-1} + h_1$, $\forall x \in \bar{\Omega}$, $\xi \in R^{1+m}$, where $k_1 > 0$ and $h_1 \in L^q(\Omega)$;

(C_2) There exist a continous function $k : R_+ \mapsto R_+$with the property that $k(t)\,t^{-s} \to 0$ when $t \to \infty$ sach that, $|C(x,\xi(u))| \leq k\left(\|u\|_{1,p}\right)$ for $u \in H^{1,p}(\Omega)$.

We will show that this hippotheses are sufficient for the existens of generalized solution of problem (1).

The hippotheses (A_1) and (C_1) we assure the Dirichlet forms:

$$a(u,v) := \sum_{|\alpha|\leq 1} \int_\Omega A_\alpha(x,\ \xi(u)) D^\alpha v\, dx$$
$$b(u,v) := \int_\Omega C(x,\ \xi(u))\, v\, dx$$

are well defined for $\forall u, v \in H^{1,p}(\Omega)$ (where $H^{1,p}(\Omega)$ is a reflexive Banach space and separable); there are continous functioning of $v \in H^{1,p}(\Omega)$, for $u \in H^{1,p}(\Omega)$ arbitrar fixed wich is result from inequality of Hölder. It result that we can attach the problem (1) of functional equation

$$(2) \qquad a(u,v) + b(u,v) = (f,v), \forall v \in H^{1,p}(\Omega)$$

with $f \in L^q(\Omega)$ given, wich the solution we call (weak) generalized solution of problem (1).

Because $A(u) := a(u,.)$ and $C(u) := b(u,.)$ are functionable on $X : = H_0^{1,p}(\Omega)$ we can defined bounded and continu operator, $A : X \mapsto X^*$ and $C : X \mapsto X^*$, by

$$(Au,\ v) := a(u,v),\ \forall v \in X$$
$$(Cu,\ v) := b(u,v),\ \forall v \in X,$$

where $(Au,\ v)$ and $(Cu,\ v)$ reprezented $A(u)$ and $C(u)$ from X^* wich is calculated in $v \in X$.

Formach $f \in L^q(\Omega)$ there exist an $\omega_f \in X^*$ unique determinate for equation $(\omega_f, v) = (f, v)\,, \quad \forall v \in X$.

The equation (2) is equivalent with operatorial equation

$$(3) \qquad T(u) := Au + Cu = \omega_f,$$

determinated by an $\omega_f \in X^*$ known and the operator $T := A + C : X \mapsto X^*$.

For the existens of solution (3) we applied the theory of applications A-proper; must bild a projective scheme complet for application from $X \text{in} X^*$.

Let $E : = \{e_1,\ ...,\ e_n,\ ...\}$ a system of linear independents elements, complet in X. We defined subspaces finit dimensional

$$X_n : = Sp\{e_1,\ ...,\ e_n,\ ...\}$$

such that $\bigcup\{X_n; n \in N\}$ let be a dense set in X. For each $n \in N$ we associed the equation (2) an algebric system, of $n$nonlinear equation,

$$(4) \qquad a(u_n, e_j) + b(u_n, e_j) = (f, e_j)\,, \quad 1 \leq j \leq n,$$

with determeins approximate solution, $u_n : = a_1^{(n)} e_1 + ... + a_n^{(n)} e_n$. We denote with $P_n : X \mapsto X_n$ the linear projection induse by system $E$and with $I_n^{:} X_n \mapsto X$linear injection of $X_n \text{in} X$. Because $Y := X*$, we take $Y_n : = X_n*$ and we define the projection $Q_n : Y \mapsto Y_n$ like adjoined application of injectiviteI_n, then $Q_n : = I_n*$.

$\Gamma_n : = (\{X_n\}\,, \{Y_n\}\,, \{P_n\}\,, \{Q_n\})$ mades a complet projective of approximation scheme; but equation (4) is equation equivalent

$$(5) \qquad T_n(u_n) : = A_n u_n + C_n u_n = Q_n \omega_f$$

where $A_n : = Q_n A / x_n, \quad C_n : = Q_n C / x_n$ and $T_n : = A_n + C_n :\ X_n \mapsto Y_n$.

Lemma: *In hypotheses ($A_1 - A_2$) and (C_1), the operators A and T : =A + C are A-proper with respect to the scheme Γ_n (then T : =A + C is an (weak) operator A-proper like the sum from an operator A weak A-proper and an compact operator C.*

Theorem: *In hypotheses (A) and (C) there exist at last a generalist solution of a problem (1), which can be obtain by aproxing with the scheme Γ_n.*

Proof: In the base of Lemma, the operators$A$and T : =A + C are A-proper with respect to scheme Γ_n.

A is odd and omogen positive of order s¿0, in the base of hypotheses (A_3), but Chas the properties

$$(6) \qquad \frac{\|Cx\|}{\|x\|^S} \to O \text{ when } ||x|| \to \infty,$$

in the base of hypotheses (C_3).

The set $B_n : = P_n(B(O,r))$ is bounded and open in X_n for each $n \in N$, but $\bar{B}_n$ and ∂B_n are closed in $P_n(\bar{B}(O,r))$.

For each $n \in N$ we consider the homotopic continuous

$$H_n(x,t) := A_n x + (1-t)(C_n x - Q_n \omega_f), \quad \forall x \in \bar{B}_n,\ t \in [0,1]$$

and we assert that there exist an $r > 0$ and a rank $N \geq 1$ such that $O \notin H_n(\partial B_n,\ [0,1])$, $\forall n \geq N$.

If it wasn't like that we can fined the row $\{x_m;\ x_m \in X_m\}$ and $\{t_m\} \subset [0,1]$, sach that $||x_m|| \to \infty$ and

$$Q_m A x_m + (1-t) Q_m (C x_m - \omega_f) = O, \quad \forall m \in N. \tag{7}$$

Because Q_m are linear uniformly bounded application, A is omogen positive of order $s > 0$ and $\|x_m\| \to \infty$, from (6) and (7) follows that

$$Q_m A\, z_m = \frac{(t_m - 1) Q_m (C x_m - \omega_f)}{\|x_m\|^S} \to 0, \text{where}$$

$$z_m := \|x_m\|^{-1} x_m$$

Because $||z_m|| = 1$ and $A_m z_m := Q_m A\, z_m \to 0$, we used the fact that A is an operator A-proper, it result that there exist an subrow $\{z_i\} \subset \{z_m\}$ and $z \in X$, with $\|z_m\| = 1$, such that $z_i \to z$ and $Az = 0$, are in contradiction with the propretes of A.

So, we can use the invariation to the homotopic of Brouwer degees and, because A_n is odd for each $n \in N$, using the Borsuk theorem, we obtain

$$d(T_n,\ B_n,\ Q_n \omega_f) =: d(H_n(.,\ 0),\ B_n,\ O) =$$

$$= d(H_n(.,\ 1),\ B_n,\ O) := d(A_n,\ B_n,\ O) \neq 0.$$

Because T is a weak operator, the generalized topological degree is well defined and it results from up $D(T,\ B,\ \omega_f) \neq \{0\}$; we deduse that there exist at last one solution $x \in B(O,r)$ of equation (3), which can be obtaine like weak limit (is hard T if is injectiv), of a subrow from the row of approximate equations solutions (5).

2. The abstract Hammerstein equation.

Let $A : X^* \mapsto X$ an compact and negative linear operator, but $N : X \mapsto X^*$ is an bounded and semicontinu operator, where X and X^* is a Banach space and his dual. If is satisfaied the corectivity condition $(Nu,\ u - \omega) \geq 0$ for all $u \in \partial D$, with D is an open and bounded set in X, and a certain $\omega \in D$, then Hammerstein equation $(I + AN)\, u = \omega$ has at last one solution in $\bar{D}$.

Proof: we consider the perturbation

$$N_\varepsilon u := N\, u + \varepsilon\, J(u - \omega)$$

where $J : X \mapsto X*$ is the duality application

$$J\, x := \left\{ f \in X*;\ (f,x) = \|x\|^2 = \|f\|^2 \right\}$$

(for example Pascali – Sburlan [1, p. 109]); then N_ε has the same properties like N then the operator $P_\varepsilon := A\, N_\varepsilon : X \mapsto X$ is compact.

We show, first, with the continuotion method that the equation $(I + P_\varepsilon) U = \omega$ admit a solution $u_\varepsilon \in D$. For this we most shown that the equation $(I + t\, P_\varepsilon) U = \omega$ has not solutions on ∂D for any $t \in [0,1]$.

We suppose, by absurd, for $t \in [0,1]$ there exist a solution $u_\varepsilon \in \partial D$ of this equation; then

$$0 = (N_\varepsilon u,\ \ u + t\, P_\varepsilon u - \omega) = (N_\varepsilon u,\ \ u - \omega) + t\,(N_\varepsilon u,\ \ A\, N_\varepsilon u)$$

Because A is nonnegative, $(N_\tau u, A N_\tau u) \geq 0$ it result

$$0 \geq (N_\varepsilon u,\ \ u - \omega) = (Nu,\ \ u - \omega) + \varepsilon\,(J(u - \omega),\ \ u - \omega) \geq \|u - \omega\|^2 .$$

So $\omega = u \in \partial D$, in contradiction with what we had supposed.

In consequence, for each $\varepsilon > 0$, there exist a solution $u_\varepsilon \in D$ of equation $(I + P_\varepsilon)\, U = \omega$. More then

$$(I + A\,N)\; u_\varepsilon = \omega - \varepsilon\, A\, J\left(u_\varepsilon^{-} \omega\right) =: \; \omega_\varepsilon.$$

From this equality it results that $\omega_\varepsilon \to \omega$ in X, when $\varepsilon \to 0$, so ω is in the set $(I + AN)\,(D)$. Because the product $A\,N$ is an compact operator, we have $\omega \in (I + A\,N)\left(\bar{D}\right)$, so what we have to proof.

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