ON THE REPRESENTATION OF A $H^1(a,b)$ DISTRIBUTION THAT FULFILS $u(a)=0$

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ABSTRACT. We define the closed cone of increasing functions and represent each u as a sum of its projection on cone and the remaining part. We then define a scalar product and prove that the induced norm convergence implies the uniform convergence on [a,b].

Proposition 1. *The set of the increasing function on* $V \overset{not}{=} H^1(0,1)$ *with* $u(0)=0$ *is a closed con convex point zero.*

Proof :

Con : If $v_1, v_2 \in C$ *then* $v_1+v_2 \in C$; $v \in C, \alpha \geq 0$ *then* $\alpha v \in C$.

Convex con : $\alpha v_1 + (1-\alpha) v_2 \in C$, for any $v_1, v_2 \in C$, and any $\alpha \in (0,1)$.

If v_1, v_2 increasing with $v_1(0)=v_2(0)=0$ impliesv_1+v_2 with $(v_1+v_2)(0)=0$. If$\alpha \succ 0$, v increasing, implies increasing αv with $(\alpha v)(0)=0$.

Closed con : If $f_n(x) \overset{V}{\longrightarrow} f(x)$ *implies*$f_n(x) \overset{punctual}{\longrightarrow} f(x)$, for any $x \in [0,1]$.

How f_n increasing, $f_n \overset{punctual}{\longrightarrow} f$ implies f increasing.

$$f_n(x_2) - f_n(x_1) \geq 0 \lim_{n\to\infty} (f_n(x_2) - f_n(x_1)) = f(x_2) - f(x_1) \geq 0$$

In particular: $f_n \in C$, $f_n \overset{V}{\longrightarrow} f$ implies $f_n \overset{punctual}{\longrightarrow} f$ and how f_n increasing implies f increasing and $f \in C$ implies $C = \overline{C}$.

Proposition 2. *Let* $u^* \in V$*(*V *gifted with* $a(u.v) = \int_0^1 u(x)' v(x)' dx$*). Then his projection* u^* *in given norm* $\left(\|u\| = \sqrt{a(u,u)} = \sqrt{\int_0^1 u(x)'^2 dx} \right)$ *on the con exist is unique and satisfied the relation (see Philippe Ciarlet[1]): there exist* $u_1 \in C$ *,* u_1 *increasing such that*

$$a(u_1, v_h) \geq a(u^*, v_h), \textit{for any } v_h \in C$$

$$a(u_1, u_1) = a(u^*, u_1).$$

Proof :

$$a(u^* - u_1, v_h) \leq 0$$

$$\int_0^1 (u^*(x) - u_1(x))\prime v_h(x)\prime dx \leq 0, \quad \forall v_h \in C, v_h' \geq 0$$

implies

$$(u^*(x) - u_1(x))' \leq 0$$

$$u_2(x) \overset{not}{=} u^*(x) - u_1(x)$$

that is u_2 decreasing; $C_1 = \{u \in V, u \ decreasing\} = -C$.

Writing $u^* = u_1 + u_2$ with u_1 increasing, $u_1(0)=0$ and u_2 decreasing $u_2(0)=0$.

Proposition 3. *We have put* $p(u,v) = \int_0^1 (u_1(x)' v_1(x) + u_2(x)' v_2(x))dx$; *with* $u = u_1 + u_2$, $v = v_1 + v_2$ *discomposing after C, where* $p(u,v)$ *is linear in each argument, strictly positive.*

Proof :

Let $\overline{u} = \overline{u}_1 + \overline{u}_2$, where $\overline{u_1 + u_2} \in C$ increasing and $\overline{\overline{u}} = \overline{\overline{u}}_1 + \overline{\overline{u}}_2$ where $\overline{\overline{u_1 + u_2}} \in -C$ decreasing.

$$p\left(\overline{u} + \overline{\overline{u}}, v\right) = \int_0^1 (\left(\overline{u}_1(x) + \overline{\overline{u}}_1(x)\right)' v_1(x) + \left(\overline{u}_2(x) + \overline{\overline{u}}_2(x)\right)' v_2(x))dx =$$

$$= \int_0^1 \left(\overline{u}_1(x) + \overline{\overline{u}}_1(x)\right)' v_1(x)\, dx + \int_0^1 \left(\overline{u}_2(x) + \overline{\overline{u}}_2(x)\right)' v_2(x)\, dx =$$

$$= \int_0^1 \overline{u}_1(x)' v_1(x)\, dx + \int_0^1 \overline{\overline{u}}_1(x)' v_1(x)\, dx + \int_0^1 \overline{u}_2(x)' v_1(x)\, dx + \int_0^1 \overline{\overline{u}}_2(x)' v_1(x)\, dx.$$

implies the linearity u,similarly in v; than $p(u,v)$is linear in each argument.

$$p(u,u) = \int_0^1 ((u_1(x))' u_1(x) + (u_2(x))' u_2(x))dx =$$

$$= \frac{1}{2}\int_0^1 (\left(u_1(x)^2\right)' + \left(u_2(x)^2\right)')dx = \frac{1}{2}\left(u_1^2(1) + u_2^2(1)\right) \geq 0 \quad .$$

$p(u,u) = 0$ impies $u = 0$.

Proposition 4. *We put* $b(u,v) = p(u,v) + p(v,u)$ *a scalar product on V, where*

$$\|u\|_b = \sqrt{b(u,u)}, \quad b(u,u) \geq 0$$

$$\left(p(u,u) \geq 0, p(u,u) = \int_0^1 ((u_1(x))' u_1(x) + (u_2(x))' u_2(x))dx\right).$$

Proposition 5. $u_n \xrightarrow{\|u\|_b} u$ implies $u_n \xrightarrow{uniform} u$.

Proof :

$$\sqrt{p(u,u)} = \sqrt{\int_0^1 (u_1(x))' u_1(x)\, dx + \int_0^1 (u_2(x))' u_2(x)\, dx} =$$

$$= \sqrt{u_1^2(1) + u_2^2(1)} \quad \geq \quad \max\ \{|u_1(1)|\ ;\ |u_2(1)|\} \geq$$

$$\geq \frac{|u_1(1)| + |u_2(1)|}{2} \geq \frac{|u_1(t)| + |u_2(t)|}{2} \quad for\ any\ t$$

$$\geq \frac{|u_1(t) + u_2(t)|}{2} \geq \frac{1}{2}\|u\|_\infty$$

$$p(u,u) \geq \|u\|_\infty^2,$$

than $u_n \xrightarrow{\|u\|_b} u$ implies $u_n \xrightarrow{uniform} u$.

Proposition 6. $\|u\|_b^2 \leq K^2 \|u\|_a^2$

Proof: From inequality of the F. Poincaré we gave $\int_0^1 (u(x))^2 dx \le K^2 \int_0^1 \left(u(x)'\right)^2 dx$.

$$\int_0^1 (u_1(x))^2 dx \le K^2 \int_0^1 \left(u_1(x)'\right)^2 dx$$

$$\int_0^1 (u_2(x))^2 dx \le K^2 \int_0^1 \left(u_2(x)'\right)^2 dx$$

then

$$p(u,u) = \left| \int_0^1 ((u_1(x))' u_1(x) + (u_2(x))' u_2(x)) dx \right| \le$$

$$\le \int_0^1 \left|(u_1(x))' u_1(x) + (u_2(x))' u_2(x)\right| dx \le$$

$$\le \int_0^1 \left(\left|(u_1(x))' u_1(x)\right| + \left|(u_2(x))' u_2(x)\right|\right) dx \le \qquad (how\ \ (|a| - |b|)^2 \ge 0)$$

$$\le \int_0^1 \left(\frac{\left(u_1(x)'\right)^2 + (u_1(x))^2}{2} + \frac{\left(u_2(x)'\right)^2 + (u_2(x))^2}{2} \right) dx \le$$

then

$$\|u\|_\infty^2 \le p(u,u) \le K^2 \cdot a(u,u)$$
$$\|u\|_\infty^2 \le \|u\|_p \le K^2 \|u\|_a \quad .$$

Conclusion:

$$u_n \xrightarrow{V} u \quad that\ is\ u_n \xrightarrow{\|u\|_a} u \quad implies \quad u_n \xrightarrow{\|u\|_b} u \quad implies \quad u_n \xrightarrow{uniforma} u.$$

References

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