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# CONSTRUCTIVE SOLUTION OF PROBLEMS IN MECHANICS OF CONTINUA

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ABSTRACT. It is well known the importance of the effective solution, both analytically or numerically, of problems in mechanics. In this work we extend the multiple orthogonal sequence method to the energetic space of an abstract linear monotone operator. The method leads to an abstract eigenvalue problem that it produces orthonormal bases in some nested Hilbert spaces, that they are suitable to develop abstract Fourier or Galerkin-projection methods. Some examples are given and the constructive solution of the dynamical problem in linear elasticity is shown.

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Let $X$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. Consider a linear operator $B : D(B) \subset X \to X$, with $D(B)$ infinite dimensional, which is *symmetric*, *i.e.*

(1) $$(Bu, v) = (u, Bv), \forall u, v \in D(B)$$

and *strongly monotone*, that is, there exists $c > 0$ such that

(2) $$(Bu, u) \geq c\|u\|^2, \forall u \in D(B).$$

We induce on $D(B)$ the *energetic inner product*

$$(u, v)_E := (Bu, v), \forall u, v \in D(B)$$

and the *energetic norm*

$$\|u\| := \sqrt{(u, u)_E}, \forall u \in D(B).$$

Denote by $E$ the completion in $X$ of the linear subspace $D(B)$ with respect to the energetic norm and call it the *energetic space* of the operator $B$. It contains all $u \in X$ that are limit points of Cauchy sequences $\{u_n\} \subset D(B)$ with respect to the energetic norm $\|\cdot\|_E$. Extending by continuity the energetic inner product the energetic space $E$ becomes a real Hilbert space containing $D(B)$ as a dense subset and the embedding $E \hookrightarrow X$ is continuous, namely

$$\| u\| \leq c^{-\frac{1}{2}} \|u\|_E, \forall u \in E.$$

The duality map $J : E \to E^*$, defined through

$$< Ju, v >= (u, v)_E, \forall u, v \in E,$$

is a linear homeomorphism with

$$\|Ju\| = \|u\|_E, \forall u \in E,$$

(see D. Pascali and S. Sburlan, [7, p. 112]), and it is an extension of $B$, *i.e.*,

$$Ju = Bu, \forall u \in D(B).$$

The *Friedrichs extension*$A : D(A) \subseteq X \to X$ of operator $B$ is defined through

(3) $$Au := Ju, \forall u \in D(A),$$

where $D(A) := \{u \in E \mid Ju \in X\}$. Observe that $u \in D(A)$ if and only if there exists a $f \in X$ such that

$$< Ju, v >= (f, v)_E, \forall v \in E$$

and $D(B) \subseteq E \subseteq X \subseteq E^*$ (see E. Zeidler [11, p. 280]).

Remark that the Friedrichs extension is in fact the maximal monotone extension of $B$ in $X$, since $D(A)$ is dense in $X$ and $A$ is closed, self-adjoint, bijective and strongly monotone, *i.e.*,

$$(Au, u) \geq c \|u\|^2, \forall u \in D(A),$$

(see A. Haraux [2, p.48]). Also, the inverse operator $A^{-1} : X \to X$ is linear continuous self-adjoint and compact, whenever the embedding $E \hookrightarrow X$ is compact. Therefore applying the Fredholm theory, we can state following variant of multiple orthogonal sequence theorem (G. Moroşanu and S. Sburlan [3]):

**Theorem 1.** *If the embedding $E \hookrightarrow X$ is compact, then there exist the sequences $\{e_n\}_{n\in\mathbf{N}} \subset E$ and $\{\lambda_n\}_{n\in\mathbf{N}} \subset (0,\infty)$ that are eigensolutions of $A$, i.e.,*

$$(Ae_n, v) = \lambda_n(e_n, v), \forall v \in X, n \in \mathbf{N} \tag{4}$$

*and such that:*

*i) $\{e_n\}_{n\in\mathbf{N}}$ is an orthonormal basis in $E$;*
*ii) $\{\sqrt{\lambda_n}e_n\}_{n\in\mathbf{N}}$ is an orthonormal basis in $X$;*
*iii) $\{\lambda_n e_n\}_{n\in\mathbf{N}}$ is an orthonormal basis in $E^*$;*
*iv) $\{\lambda_n\}_{n\in\mathbf{N}}$ is increasingly divergent to $+\infty$.*

**Direct consequence:** Denote by

$$E_n := Sp\{e_1, e_2, \ldots, e_n\} \subset E,$$

$$X_n := Sp\{\sqrt{\lambda_1}e_1, \sqrt{\lambda_2}e_2, \ldots, \sqrt{\lambda_n}e_n\} \subset X$$

and

$$E_n^* := Sp\{\lambda_1 e_1, \lambda_2 e_2, \ldots, \lambda_n e_n\} \subset E^*,$$

the finite dimensional subspaces generated by the finite sequence $\{e_1, e_2, \ldots, e_n\}$.

Then $E_n$, $X_n$ and $E_n^*$ are *projectionally complete* in $E$, $X$ and $E^*$ respectively, that is $\pi_n u \to u$ in each space, where

$$\pi_n u := \sum_{k=1}^{n} \alpha_k \varphi_k, \forall n \geq 1, \tag{5}$$

with $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ one of the above mentioned basis and $\alpha_k, 1 \leq k \leq n$, the corresponding Fourier coefficients.

These coordinate systems can be used either for abstract Galerkin projection method or for abstract Fourier series method (see S. Sburlan and G. Moroşanu [10]).

Consider the following Cauchy problem in $X$:

$$\begin{cases} y''(t) + Ay(t) = f(t), 0 \leq t \leq T \\ y(0) = y_0, y'(0) = y_1, \end{cases} \tag{6}$$

under the following assumptions on the data

$$y_0 \in E, y_1 \in X, f \in L^2(0,T;X). \tag{7}$$

Searching the solution of the form

$$y(t) = \sum_{n=1}^{\infty} b_n(t) e_n, \tag{8}$$

we obtain that the coefficients must satisfy the equations

$$\begin{cases} b_n''(t) + \lambda_n b_n(t) = f_n(t), 0 \leq t \leq T \\ b_n(0) = y_{0n}, b_n'(0) = y_{1n}, \end{cases} \tag{9}$$

where the Fourier coefficients

$$y_{0n} =< y_0, e_n >_E = \lambda_n(y_0, e_n), \;\; f_n(t) :=< f(t), e_n >= \lambda_n(f(t), e_n)$$

and

$$y_{1n} := (y_1, e_n)_E = \lambda_n(y_1, e_n).$$

After solving (9), we obtain

$$b_n(t) = y_{0n} \cos \sqrt{\lambda_n} t + \frac{y_{1n}}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \int_0^t f_n(s) \sin \sqrt{\lambda_n}(t-s) ds. \tag{10}$$

and it is true the following:

**Theorem 2.** (Moroşanu-Sburlan [5]) *Under the hypotheses (7), the function $y(t)$ given by (8), (10), belongs to $C([0,T];E) \cap C^1([0,T];X) \cap H^2([0,T];E^*)$ and it is the unique weak solution of the problem (6), i.e. $y(0) = y_0, y'(0) = y_1$ and*

$$(y''(t), v) + (Ay(t), v) = (f(t), v), \forall v \in E.$$

*Furthermore, under the stronger assumption on the data*

$$f \in H^1([0,T];X), \;\; y_0 \in E, \;\; y_1 \in E, \;\; Ay_0 \in X, \tag{11}$$

*the solution $y \in C^1([0,T];E) \cap C^2([0,T];X)$ verifies classically the following Cauchy problem*

$$\begin{cases} y''(t) + By(t) = f(t), 0 \le t \le T \\ y(0) = y_0, y'(0) = y_1. \end{cases} \tag{12}$$

**Proof.**(Sketch) We obtain easily the estimations

$$|b_n(t)|^2 \le 3 \left( y_{0n}^2 + \frac{y_{1n}^2}{\lambda_n} + \frac{T}{\lambda_n} \int_0^t |f_n(s)|^2 ds \right)$$

and

$$\frac{\left| b_n'(t) \right|^2}{\lambda_n} \le 3 \left( y_{0n}^2 + \frac{y_{1n}^2}{\lambda_n} + \frac{T}{\lambda_n} \int_0^T |f_n(t)|^2 ds \right)$$

which assure that $y \in C([0,T];E) \cap C([0,T];X)$. On the other hand, by the first equation in (9) we have

$$b_n''(t) = -\lambda_n b_n(t) + f_n(t), a.a. \, t \in [0,T]$$

which leads to the estimations

$$\frac{|b_n''(t)|^2}{\lambda_n^2} \le C \left( |b_n(t)|^2 + \frac{|f_n(t)|^2}{\lambda_n} \right), C > 0,$$

$$\frac{|b_n''(t)|^2}{\lambda_n} \le 4 \left( \lambda_n y_{0n}^2 + y_{1n}^2 + \frac{|f_n(0)|^2}{\lambda_n} + T \int_0^T \lambda_n \left| f_n'(s) \right|^2 ds \right).$$

These estimations are used to prove either $y'' \in L^2([0,T];X)$ in the case of weak solution, or $y'' \in C([0,T];X)$ in the case of classical solution. □

The above conditions can be extended to semilinear problems when $f(t)$ is perturbed by a small nonlinearity, namely

$$f(t,y) := f(t) + \delta g(t,y) \tag{13}$$

where $\delta > 0$ is so small that $\delta^2$ can be neglected. Indeed, the key step consists in proving some estimations concerning $b_n(t)$ and its derivates. These estimations depend on

$$|f_n(t,y)|^2 \le 2 \left( |f_n(t)|^2 + \delta^2 |g_n(t,y)|^2 \right) \le C |f_n(t)|^2,$$

with an available $C > 0$. Thus we can reduce this semilinear case to the former one studied above.

Dynamical problem in linear theory of elasticity is a relevant problem for this method:

The deformation of a body $\mathcal{B}$, that occupies a bounded region $\Omega$ in the space $\mathbf{R}^N$ ($N = 2$ or 3), is characterized by the *displacement vector* $u : \Omega \times (0,T) \to \mathbf{R}^N$ and the corresponding *strain tensor* $\varepsilon = \varepsilon(u)$. In the case of small (infinitesimal) deformation, $\varepsilon(u)$ reduces to the symmetric part of the *displacement gradient*, *i.e.*,

$$\varepsilon(u) := \left\{\varepsilon_{ij}(u) \mid \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), 1 \le i,j \le N\right\}. \tag{14}$$

The constitutive relation that characterizes the elasticity is a generally nonlinear dependence of the *stress tensor*

$$\sigma := \{\sigma_{ij} | \sigma_{ij} = \sigma_{ji}, 1 \le i,j \le N\}$$

on the strain, namely,

$$\sigma = \sigma(\varepsilon) = \mathcal{A}\varepsilon + \sigma(|\varepsilon)^2), \tag{15}$$

where $\mathcal{A} := \{a_{ijkl} \in \mathbf{R} | a_{ijkl} = a_{jikl} = a_{klij}, 1 \le i,j,k,l \le N\}$ are elastic coefficients.

Dynamical problem of linear elasticity ask for the displacement $u$, that is the solution of the initial and boundary value problem

$$\begin{cases} u_{tt} - \mathrm{div}\sigma(\varepsilon(u)) = f \;\; in \;\; \Omega \times (0,T), \\ u = 0 \;\; on \;\; \Gamma \times (0,T), \\ \sigma_{ij}(\varepsilon(u))n_j = 0 \;\; on \;\; (\partial\Omega \setminus \Gamma) \times (0,T), \\ u(0) = u_0, \;\; u_t(0) = u_1. \end{cases} \tag{16}$$

Define on

$$\begin{aligned} D(B) \;\; := \;\; & \{y \in [C^2(0,T;C^2(\bar\Omega))]^N \mid y = 0 \; on \; \Gamma \times (0,T), \\ & \sigma_{ij}n_j = 0 \; on \; (\partial\Omega \setminus \Gamma) \times (0,T)\} \end{aligned}$$

the linear operator induced by the problem (16):

$$By := -\mathrm{div}\sigma(\varepsilon(y)).$$

Then by Green's formula we have:

$$(Bu, v) := -\int_\Omega \mathrm{div}\sigma(\varepsilon(u))v ds = \frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(v)dx =$$

$$= \frac{1}{2}\int_\Omega a_{klij}\varepsilon_{ij}(v)\varepsilon_{kl}(u)dx = -\int_\Omega \mathrm{div}\; \sigma(\varepsilon(v))udx =: (u, Bv), \forall u, v \in D(B).$$

Moreover, we have

$$(Bu, u) = \frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)dx \ge \frac{c}{2}\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u)dx = c\int_\Omega |\nabla u|^2 dx,$$

by the symmetry of $\varepsilon$.

Therefore we can apply the above theory and define the energetic space $E$ as the completion of $D(B)$, with respect to the norm

$$\|u\|_E^2 := \frac{1}{2}\int_\Omega \sigma_{ij}(\varepsilon(u)) \cdot \varepsilon_{ij}(u)dx = \frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)dx$$

The duality map $J_E : E \to E^*$ is defined by

$$< J_E u, v >_E := \frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)dx,$$

and thus we obtain the Friedrichs extension as it is shown. Remark that

$$\mathcal{W} = 2 < J_E u, u >= \int_\Omega a_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)dx$$

is the strain energy induced by $u$.

In conclusion, it is true the following

**Theorem 3.** *If $y_0 \in E$, $y_1 \in X$ and $f \in [L^2(0,T;X)]^N$, then there exists a unique weak solution of the problem (16) that can be obtained by Fourier method. Furthermore, for $Ay_0 \in X$ and $f \in [H^1(0,T;X)]^N$ this solution is a classical one.*

Similar results are also true for the dynamical problem of linear theory of finte elasticity (see [13-14]).

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