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THE TOPOLOGIC STUDY OF THE GEOMETRIC TRAJECTORIES IN THE MATHEMATICAL BILLIARDS

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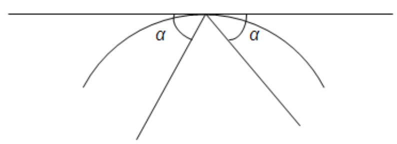
The mathematical billiard problem is about the reflection of a light ray in different bounded domains. In this paper we will refer to the domain bounded by a circle.

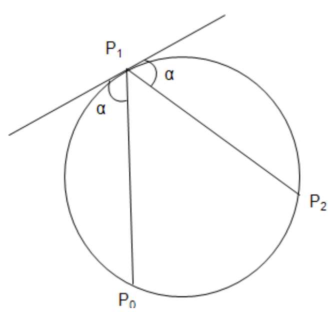
Let us consider a light ray that reflects repeatedly after touching the curve and thus it will describe a certain trajectory. The problem is what type is this trajectory and whether an envelope of these reflected rays exists.

As it is already known, the ray is reflected by an angle congruent at the tangent to the curve in the reflexion point.



The problem in the circle

Let α be the angle the ray makes with the tangent to the circle in a given point.



Let $(P_i)_{i\in\mathbb{N}}$ the contact points with the circle of the reflected ray, namely the points where the trajectory touches the circle, where P_0 is the initial position. It is obvious that

$$m\left(\widetilde{P_iP_{i+1}}\right) = 2\alpha.$$

In the study of the problem, two cases arise:

1. α is comensurable rational with π, that is:

$$\alpha = \frac{m}{n}\pi, m \in \mathbb{N}^*, m \leq \left[\frac{n}{2}\right], n \in \mathbb{N}^*\setminus\{1\};$$

2. α is not comensurable rational with π.

In the first case, if α is comensurable with π,that is:

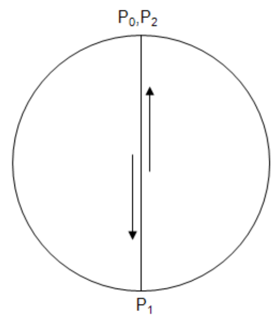
$$\alpha = \frac{m}{n}\pi \Rightarrow (m,n) = 1 \Leftrightarrow n\alpha = m\pi.$$

and obviously after n reflextions it arrives at the departure point. Thus we obtain $P_i = P_{i+n}$, since

$$m\left(\widetilde{P_k P_{k+1}}\right) = 2\alpha, \forall k \in N \Rightarrow$$
$$m\left(\widetilde{P_i P_{i+1}}\right) + m\left(\widetilde{P_{i+1} P_{i+2}}\right) + ... + m\left(\widetilde{P_{i+n-1} P_{i+n}}\right) = 2\alpha n = 2m\pi,$$

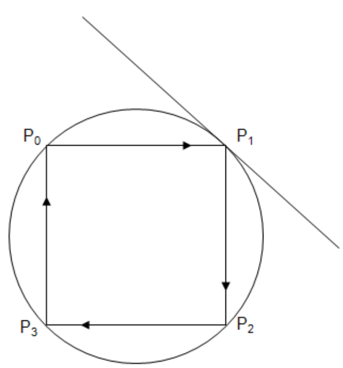
hence it follows that the trajectory is periodical.

If $m = 1, n = 2 \Rightarrow \alpha = \frac{\pi}{2}$, then $P_0 = P_2, P_1 = P_3$ and practically the trajectory is a segment.



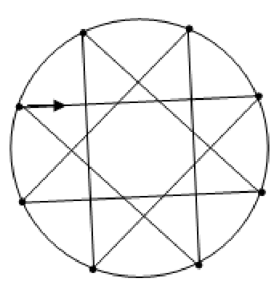
If $m = 1$ and $n \in \mathbb{N}$, $n \geq 3$, the trajectory is a regular poligon with n sides.

Example: for $\alpha = \frac{\pi}{4} \Rightarrow$



If $m > 1$ and $(m,n) = 1$, the trajectory isa closed stellar one with n points. The rays reflected are intersected themselves.

Example. $\alpha = \frac{3\pi}{8}$



In the second case (α not comensurable with π), there are several problems:

1. the trajectory is not periodical, that is $P_i \neq P_j, (\forall)\, i \neq :, i, j \in \mathbb{N}$;

2. the trajectory touches any point of the circle (any point of the circle, will be contact point of the reflected ray)

3. the reflected rays cover entirely a circular ring.

Prop 1. If $\alpha = x\pi, x \notin \mathbb{Q}$, then the trajectory is not periodical.

Dem.: By reduction to absurd we suppose thet the trajectory is periodical. It follows that:

$$(\exists)\, k \in \mathbb{N}^* a.i. P_i = P_{i+k}, (\forall)\, i \in \mathbb{N} \Rightarrow (\exists)\, m \in \mathbb{N}^* \Rightarrow$$
$$m\left(\widetilde{P_i P_{i+1}}\right) + m\left(\widetilde{P_{i+1} P_{i+2}}\right) + ... + m\left(\widetilde{P_{i+k-1} P_{i+k}}\right) = 2m\pi,$$

since we arrive at the departure point if m circle rotations

$$\Rightarrow 2k\alpha = 2m\pi \Rightarrow \alpha = \frac{m}{k}\pi \Rightarrow$$

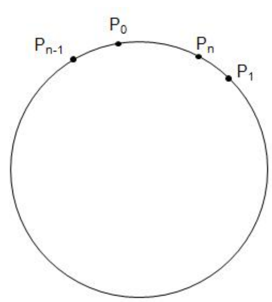
α comensurable rational with $\pi \Rightarrow$ contradiction it follows the trajectory is nonperiodical.

We will prove now that the set of the contact points isdense of the circle, so every circle point will be a contact point.

So, for any arc $\triangle$ very small, there is $(\exists)$ a point $M \in \triangle$ which is a contact point for the reflected ray.

Theorem 1: Let $\frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ and the infinite set $\{P_0, P_1, ..., P_n, ...\} = \{P_k\}$ a set of circle points so that every point P_{k+1} is obtained from P_k by a rotation around the circle with α radians. Then every arc $\triangle$ of the circle contains a point in the set $\{P_k\}$.

Dem.



Starting from P_0 with arcs of measure α, we get a nonperiodical trajectory $P_1, P_2, ..., P_n, ...$ Let $n \in \mathbb{N}$ with the property $m\left(\widetilde{P_0 P_{n-1}}\right) < 2\pi$ and $m\left(\widetilde{P_0 P_n}\right) > 2\pi$. Since the trajectory is nonperiodical $\Rightarrow m\left(\widetilde{P_0 P_n}\right) \neq 2\pi$.Putting $A_1 = P_0$ and considering A_2 the proximate point from A_1 between P_{n-1} and P_n we denote and $\alpha_1 = m\left(\widetilde{A_1 A_2}\right)$ we have $\alpha_1 \leq \frac{\alpha}{2}$.

We continue the movement from A_2 by rotations of α - measured arcs and after a numberof steps we arrive at a point far from A_1 with an arc α_1 then $2\alpha_1, ...$ In this way we have a movement α_1 from A_1, through points that are also situated on the trajectory $(P_i)_{i\in\mathbb{N}}$. Since the trajectory is nonperiodical, $\exists\, k, j$ so that $A_1 \in \widehat{P_k P_j}, m\left(\widehat{P_k P_j}\right) = \alpha_1$.

We chose A_3 the proximate point from A_1, between P_k and $P_j \Rightarrow m\left(\widehat{A_1 A_3}\right) = \alpha_2 \leq \frac{\alpha_1}{2} \leq \frac{\alpha}{2^2}$.

Similarly, we arrive at the arc $\widetilde{A_1 A_n}$ with $m\left(\widehat{A_1 A_n}\right) = \alpha_{n-1} \leq \frac{\alpha}{2^{n-1}}$.

Since

$$\lim \frac{\alpha}{2^{n-1}} = 0 \Rightarrow (\forall)\, \varepsilon > 0, (\exists)\, n_\varepsilon a.i. \frac{\alpha}{2^{n-1}} < \varepsilon, (\forall)\, n > n_\varepsilon$$

$$\Rightarrow (\exists)\, i, j \in \mathbb{N}.a.i.m \left(\widehat{P_i P_j}\right) < \varepsilon$$

$\Rightarrow$ $\exists$ a point $P_k \in \triangle$.

In conclusion, we have proved that on every arc $\triangle$, $\exists$ a point P_k of the reflacted trajectory.

Theorem 2. On every arc $\triangle$ there is an infinity of points P_i of the trajectory.

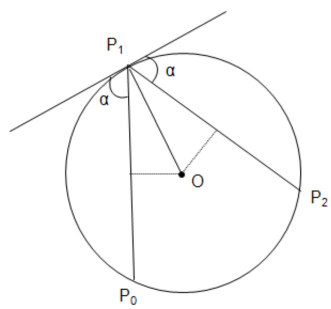
Dem. We consider there are only a finite number of points on an arc $\triangle$. We denote these points by P_{i_1},....., P_{i_n} in this order.

Denoting $\triangle_1 = \widehat{P_{i_1} P_{i_2}} \Rightarrow (\exists)$, there is a point on this circle and it follows a contradiction.

According to theorem 1 we have the density of $\{P_k\}$ on the circle.

For the third problem we consider that the ray leaves a coloured trace with no thiekness. We wonder what portion of the circle is coloured.

To anower this question, we shou first fall that the envelope of the reflacted rays is a circle γ concentric with the initial circle Γ, with the ray $r = d(O, P_0P_1)$, where O is the circle centre.



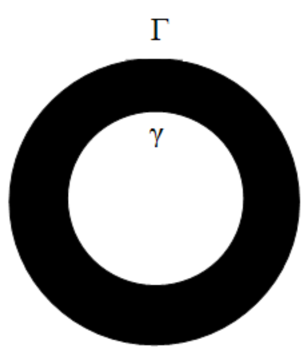
P_1O is the bisector $\angle P_0P_1P_2 \Rightarrow$

$$d\,(O, P_1P_2) = d\,(O, P_0P_1) \Rightarrow d\,(O, P_kP_{k+1}) = d\,(O, P_0P_1) = r$$

it follows that the reflected rays will be tangent to the circle γ.

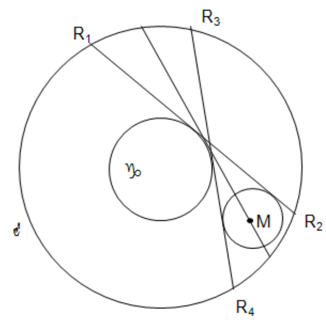
In conclusion, the envelope of these reflected rays is the circle γ.

Theorem 3. If we consider that the ray leaves a coloured nonthick troce, then it will cover densely the circular ring between the circles Γ and γ, denoted by K.



Dem. Supposing $(\exists)\, M \in K$ so that $(\forall)\, k \in \mathbb{N}, M \notin P_kP_{k+1} \Rightarrow (\exists)\, \varepsilon > 0$ $\Rightarrow$ so that the circle centred in M and with the ray ε, denoted by C, is included in K and it is not intersected by any trajectory P_kP_{k+1}.

We put R_1R_2 and R_3R_4 the commun tangents to the two circles as shown below.

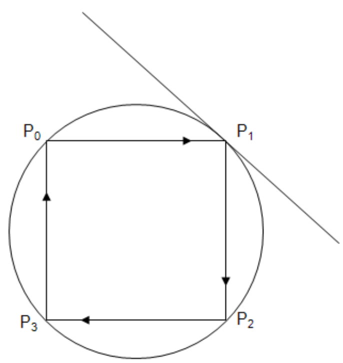


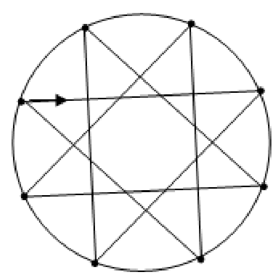
Since on every arc there is a point on the trajectory $\{P_k\}_{k\in\mathbb{N}} \Rightarrow (\exists)\, P_k \in R_1R_3 \Rightarrow P_kP_{k+1}$ will be the tangent to γ and will intersect the circle C, so there is a contradiction.

Therefore the ring K is densely covered by the segment P_iP_{i+1}.

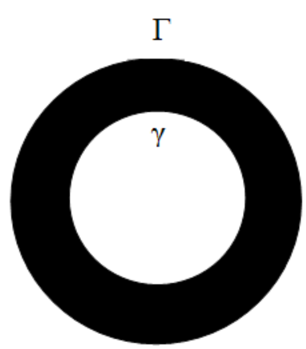
We conclude that in the mathematical billiards in a circle, there ere the following cases:

1. the trajectory is a regular poligon or a star figure, if the angle made by the first ray with the tangent to the circle in the intersection point of the ray with the circle is comensurable with π.





2. the trajectory covers densely a circular ring, otherwise.



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