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ON AN INTEGRAL INEQUALITY

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Abstract. In this note we will present two proofs for an integral inequality. The first uses the Jensen's Inequality and the second uses ingeniously the Cauchy-Schwartz Inequality.

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The Result

Theorem 1 Let $f:[0,1]\to(-1,1)$ be a continuous function so that $\int\limits_0^1 f(x)dx \notin \{-1,1\}$. Then:

$$\frac{\left(\int\limits_0^1 f(x)dx\right)^2}{\sqrt{1-\left(\int\limits_0^1 f(x)dx\right)^2}} \leq \int\limits_0^1 \frac{f^2(x)}{\sqrt{1-f^2(x)}}dx. \tag{1}$$

Proof 1 : We will use the following

Theorem(Jensen's Inequality) (see [1]) Let $f:[0,1]\to(u,v)$ be a continuous function and $g:(u,v)\to R$ be a convex function. Then

$$g\left(\int\limits_0^1 f(x)dx\right) \leq \int\limits_0^1 g(f(x))dx.$$

Let $g:(-1,1)\to R$ defined by $g(x)=\frac{x^2}{\sqrt{1-x^2}}$. We have $g\in C^2(-1,1)$ and

$$g'(x)=\frac{2x-x^3}{(1-x^2)\sqrt{1-x^2}}$$

$$g''(x)=\frac{x^2+2}{(1-x^2)^2\sqrt{1-x^2}}.$$

It is clear that $g''(x)\geq 0$ on (-1,1) and, consequently, g is a convex function. The inequality (1) results now with the Jensen's Inequality.

Proof 2 : With the Cauchy-Schwartz inequality we obtain

$$\left(\int\limits_0^1 f(x)dx\right)^2 = \left(\int\limits_0^1 \frac{f(x)}{\sqrt[4]{1-f^2(x)}}\sqrt[4]{1-f^2(x)}dx\right)^2 \leq$$

$$\left(\int\limits_0^1 \frac{f^2(x)}{\sqrt{1-f^2(x)}}dx\right)\left(\int\limits_0^1 \sqrt{1-f^2(x)}dx\right).$$

Thus

(2) $$\left(\int_0^1 f(x)dx\right)^2 \le \left(\int_0^1 \frac{f^2(x)}{\sqrt{1-f^2(x)}}dx\right)\left(\int_0^1 \sqrt{1-f^2(x)}dx\right).$$

We apply one more time the Cauchy-Schwartz inequality and we have

$$\left(\int_0^1 \sqrt{1-f^2(x)}dx\right)^2 = \left(\int_0^1 \sqrt{(1+f(x))(1-f(x))}dx\right)^2 \le$$

$$\le \left(\int_0^1 (1+f(x))dx\right)\left(\int_0^1 (1-f(x))dx\right) = \left(\int_0^1 dx\right)^2 - \left(\int_0^1 f(x)dx\right)^2 =$$

$$1 - \left(\int_0^1 f(x)dx\right)^2.$$

Therefore

(3) $$\int_0^1 \sqrt{1-f^2(x)}dx \le \sqrt{1-\left(\int_0^1 f(x)dx\right)^2}.$$

From (2) and (3) it results that

$$\left(\int_0^1 f(x)dx\right)^2 \le \left(\int_0^1 \frac{f^2(x)}{\sqrt{1-f^2(x)}}dx\right)\sqrt{1-\left(\int_0^1 f(x)dx\right)^2}$$

and the proof of the inequality (1) is complete.

References

[1] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, Inc., 1987

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