JOURNAL OF SCIENCE AND ARTS

A MOKOBODZKI TYPE THEOREM

CORNELIU UDREA

ABSTRACT. This work deals with the nonlinear potential theory actually with the corresponding specific order. Thus we shall propose a definition of the specific order ([3]) with respect to a nonlinear operator (respectively to a nonlinear resolvent) and afterwards, as in the linear case, we shall prove a Mokobodzki type theorem in the case of a nonlinear operator (respectively of a nonlinear resolvent).

1. Introduction

In this work (E,Σ, m) is a measure space (i.e. Σ is a σ-algebra of subsets of E and m is a complete measure on Σ). Moreover we suppose that m is a σ-finite measure. All functions are defined m-a. e. on E and all inequalities are accomplished m-a.e.. For all $f \in \overline{\mathbb{R}}^E$ we shall use the following standard notations:

$$m_\infty(f) := \sup\{\alpha : \alpha \in \mathbb{R} \text{ and } f \geq \alpha \text{ m-a. e.}\},$$

$$M_\infty(f) := -m_\infty(-f) \text{ and } \|f\|_\infty := M_\infty(|f|).$$

We shall consider the following sets of functions on E:

$$\mathcal{M}(m) := \{f : f \in \overline{\mathbb{R}}^E \text{ and f is m-measurable function}\},$$

$$\mathcal{M}_l(m) := \{f : f \in \mathcal{M}(m) \text{ and } m_\infty(f) > -\infty\},$$

$$\mathcal{M}^+(m) := \{f : f \in \mathcal{M}(m) \text{ and } f \geq 0\},$$

$$L^\infty(m) := \{f : f \in \mathcal{M}(m) \text{ and } \|f\|_\infty < \infty\}.$$

Throughout this work the symbols T, N will denote nonlinear increasing maps on $L^\infty(m)$ with values in $L^\infty(m)$(they are called operators ([2])) and such that T and N are continuous on the increasing sequences of $L^\infty(m)$ (we shall say that T, N are increasingly continuous). Moreover $\overline{T}$ (respectively) $\overline{N}$ will denote the natural extension of T (respectively N) to the set $\mathcal{M}_l(m)$, that is for all $f \in \mathcal{M}_l(m)$:

$$\overline{T}(f) = \sup\{Tg : g \in L^\infty(m) \text{ and } g \leq f\} = \sup\{T(\inf(f, n)) : n \in \mathbb{N}\}.$$

Obviously $\overline{T}$ (and also $\overline{N}$) is increasingly continuous (on $\mathcal{M}_l(m)$).

2. The case of a nonlinear operators

For N an increasingly continuous operator on $L^\infty(m)$ we define the set of the N-supermedian functions, and a corresponding specific order for which we prove a Mokobodzki type theorem.

Definition 2.1. *Let $u \in \mathcal{M}_l(m)$.*
*(i). The function u is called N-**supermedian function** iff $\overline{N}u \leq u$.*
(ii). We shall use the following notation:

$$\mathcal{S}_N = \mathcal{S}(N) := \{u \in \mathcal{M}_l(m) : \ u \text{ is a } N\text{-supermedian function }\}.$$

Research supported by NASR of Government of Romania: CEX-D11-23, 05.10.2005.

Remark 2.2. *(i). According to the definitions, if $u \in \mathcal{M}_l(m)$, then the following statements will be equivalent: (a). $u \in \mathcal{S}_N$. (b). $\forall g \in L^\infty(m)$ such that $g \leq u$ we have that $Ng \leq u$. (c). $\forall n \in \mathbb{N}, N(\inf(u,n)) \leq u$.*

(ii). If $N0 \leq 0$, then $0 \in \mathcal{S}_N$. In another way we shall define the sequence

$$f_1 = 0, f_2 = \sup(0, N0), \forall n \in \mathbb{N}, f_{n+1} = \sup(f_n, Nf_n),$$

and the function $r = \sup_{n\in\mathbb{N}} f_n$. It is obvious that $\overline{N}r \leq r$ and so that $r \in \mathcal{S}_N$. That being so we have that $\mathcal{S}_N \neq \emptyset$.

Proposition 2.3. *We have the following assertions:*

(i). Let $u \in \mathcal{S}_N$ and $f \in \mathcal{M}_l(m)$; it follows that: (a). $\overline{N}u \in \mathcal{S}_N$. (b). If $\overline{N}u \leq f \leq u$, then $f \in \mathcal{S}_N$.

(ii). Let $u_n \subset \mathcal{S}_N$. (a). If $\inf_{n\in N} u_n \in \mathcal{M}_l(m)$, then $\inf_{n\in N} u_n \in \mathcal{S}_N$. (b). If $(u_n)_n$ is increasing, then $\sup_{n\in N} u_n \in \mathcal{S}_N$.

Proof. They are obvious. □

Remark 2.4. *The following assertions are obvious:*

(i). When N is a subadditive operator, we have that $\mathcal{S}_N + \mathcal{S}_N \subset \mathcal{S}_N$ (see [3] *for a such nonlinear operator).*

(ii). If N has the property:

$$\forall\, u \in \mathcal{S}_N \text{ and } f \in L^\infty(m),\ \overline{N}(f+u) \leq Nf + u,$$

then we shall find again that $\mathcal{S}_N + \mathcal{S}_N \subset \mathcal{S}_N$.

(iii). If N is a positive homogeneous map, then $\forall \alpha \in \mathbb{R}_+, \alpha\mathcal{S}_N \subset \mathcal{S}_N$.

Definition 2.5. *For all $f \in \mathcal{M}_l(m)$ we define $R^N f := \inf\{u \in \mathcal{S}_N : u \geq f\}$. The function $R^N f$ is called the **N-reduced function** of f.*

Proposition 2.6. *For all $f \in \mathcal{M}_l(m)$ we have the following assertions:*

(i). The map $R^N f$ is N-supermedian function.

(ii). $R^N f = \sup(f, \overline{N}(R^N f))$.

(iii). The map $R^N : \mathcal{M}_l(m) \to \mathcal{S}_N$ is an operator which is increasingly continuous.

Proof. It is similar to the linear case ([1]).

(i). We define the sequence: $f_1 := f, f_{n+1} := \sup(f_n, \overline{N}f_n), \forall n \in \mathbb{N}^*$ and the map $r := \sup_{n\in\mathbb{N}} f_n \in \mathcal{M}_l(m)$. According to the definitions:

$$\forall\, n \in \mathbb{N}^*, f_n \leq f_{n+1}, \overline{N}f_n \leq f_{n+1} \text{ and } f \leq r.$$

Therefore

$$\overline{N}r = \sup \overline{N}f_n \leq \sup f_{n+1} = r,\ r \in \mathcal{S}_N \text{ and } R^N f \leq r.$$

If $s \in \mathcal{S}_N$ is such that $f \leq s$ then for all $n \in N^*, f_n \leq s$ and $r \leq s$ that is $r = R^N f \in \mathcal{S}_N$.

(ii). We have that $\overline{N}(R^N f) \leq \sup(f, \overline{N}(R^N f)) \leq R^N f$, hence $\sup(f, \overline{N}(R^N f)) \in \mathcal{S}_N$. Since $f \leq \sup(f, \overline{N}(R^N f))$, we shall find that $R^N f \leq \sup(f, \overline{N}(R^N f)) \leq R^N f$.

(iii). Obviously R^N is an increasing map on $\mathcal{M}_l(m)$. If $(f_n)_n \subset \mathcal{M}_l(m)$ is an increasing sequence then

$$\sup_{n\in\mathbb{N}} R^N f_n \leq R^N(\sup_{n\in\mathbb{N}} f_n) \text{ and } f_n \leq R^N f_n,\ \forall n \in \mathbb{N}.\text{Therefore}$$

$$\sup_{n\in\mathbb{N}} f_n \leq \sup_{n\in\mathbb{N}} R^N f_n \text{ and, since } \sup_{n\in\mathbb{N}} R^N f_n \in \mathcal{S}_N, \text{ we have that}$$

$$R^N(\sup_{n\in\mathbb{N}} f_n) \leq \sup_{n\in\mathbb{N}} R^N f_n. \qquad \square$$

Definition 2.7. *(i). For all $f, g \in \mathcal{M}_l(m)$ we shall denote $f \ll g$ iff $f \leq g$ and $f+\overline{N}g \leq g+\overline{N}f$.*

(ii). It is immediate that the relation $\ll$ defines an order relation on the space $L^\infty(m)$ which is called the specific order with respect to N.

Theorem 2.8 (a Mokobodzki type theorem)**.** *For all $f \in \mathcal{M}_l(m)$ and $t \in \mathcal{S}_N$ such that $f \ll t$, we shall have that $R^N f \ll t$.*

Proof. If $(f_n)_{n\in N}$ is the sequence which is defined in the proof of the Proposition 6, then we shall prove that, for all $n \in \mathbb{N}, f_n \ll t$. It is obvious that $f_1 \ll t$ and if $f_n \ll t$ we shall have the inequalities: $f_n + \overline{N}t \leq t + \overline{N}f_n$, and $f_n \leq t$. Since N is an increasing map we find that $f_{n+1} \leq t$, hence:

$$f_{n+1} + \overline{N}t = \sup(f_n, \overline{N}f_n) + \overline{N}t \leq \sup(t + \overline{N}f_n, \overline{N}f_n + \overline{N}t) = t + \overline{N}f_n \leq t + \overline{N}f_{n+1};$$

therefore $f_{n+1} \ll t$.

Moreover we remark that $f_n + \overline{N}t \leq t + \overline{N}(R^N f)$, for all $n \in \mathbb{N}^*$, and so that $R^N f + \overline{N}t \leq t + \overline{N}(R^N f)$. Since it is obvious that $R^N f \leq t$, we have that $R^N f \ll t$. $\square$

Definition 2.9. *(i).If (T,N) is pair of operators such that*

$$(I - N)(I + T) = I = (I + T)(I - N),$$

then (T,N) is called a ***pair of conjugated operators*** *(where I is the identity map of $L^\infty(m)$)(in conformity with* [2], [3] *or* [5]*).*

(ii). The pair (T,N) is a pair of conjugated operators if and only if

$$T(I - N) = N \text{ and } N(I + T) = T.$$

Remark 2.10. *Let (T,N) be a pair of conjugated operators.*

(i). If $f, g \in L^\infty(m)$ are such that $f + Ng \leq g + Nf$, then $f \ll g$ since from inequality $f - Nf \leq g - Ng$ we have that $f = (I + T)(f - Nf) \leq (I + T)(g - Ng) = g$.

(ii). For all $f \in \mathcal{M}_l(m)$ such that $f \geq 0$ we have that $Tf \in \mathcal{S}_N$.

(iii). Obviously, $T0 = 0$ if and only if N0=0.

(iv). Let $\widetilde{T},\widetilde{N} : L^\infty(m) \to L^\infty(m)$ be defined by $\widetilde{T}f = Tf - T0$, respectively $\widetilde{N}f = N(f+T0) - T0$. Then $(\widetilde{T}, \widetilde{N})$ is a pair of conjugated operators and: (a). $\widetilde{N}0 = \widetilde{T}0 = 0$. (b). $\mathcal{S}_{\widetilde{N}} = \mathcal{S}_N - T0$.

Definition 2.11. *N is called a sub-Markov operator on $L^\infty(m)$ iff for all $f, g \in L^\infty(m)$ we have that*

$$\|Nf - Ng\|_\infty \leq \|f - g\|_\infty.$$

Remark 2.12. *(i). It is obvious that $N : L^\infty(m) \to L^\infty(m)$ is a sub-Markov operator if and only if*

$$\forall f, g \in L^\infty(m), \forall\alpha \in (0, \infty), f \leq g + \alpha \Rightarrow Nf \leq Ng + \alpha.$$

(ii). Suppose that (T, N) is a pair of conjugated operators on $L^\infty(m)$ such that N is a sub-Markov one. Then

(a). $T0 + \mathbb{R}_+ \subset \mathcal{S}_N$.

(b). If for each $n \in \mathbb{N}^\star$, $e_n := n + T0$ then $(e_n)_{n\in\mathbb{N}^\star} \subset \mathcal{S}_N$ and $\lim_{n\to\infty} e_n = \infty$(m-a.e.)

3. The case of a nonlinear resolvent

Throughout this section, for all $p \in [0, \infty), V_p : L^\infty(m) \to L^\infty(m)$ is a nonlinear increasingly continuous operator.

Definition 3.1. *(see [2] and also [3], [5] or [6]).*

(i). $\mathcal{V} := (V_p)_{p\in(0,\infty)}$ is called (nonlinear) resolvent (on $L^\infty(m)$) iff for all $p, q \in (0, \infty)$ we have that:

$$(I - (p - q)V_p)(I + (p - q)V_q) = I.$$

(ii). If, for all $p \in (0, \infty)$,

$$(I - pV_p)(I + pV_0) = I = (I + pV_0)(I - pV_p),$$

*then either V_0 will be called **the initial operator** of the resolvent $\mathcal{V}$, or $\mathcal{V}$ will be called **the resolvent associated** with V_0.*

*(iii). The resolvent $\mathcal{V}$ is called a **sub-Markov resolvent** if, for all $p \in (0, \infty), pV_p$ is a sub-Markov operator on $L^\infty(m)$.*

Remark 3.2. *(i). The equalities of the resolvent's definition are equivalent to the sentence: for all $p, q \in (0, \infty), ((p - q)V_q, (p - q)V_p)$ is a pair of conjugated operators (on $L^\infty(m)$) and so that to the relation:*

$$V_p = V_q(I + (q - p)V_p), \forall p, q \in (0, \infty).$$

(ii). Similarly the property: V_0 is the initial operator of the resolvent $\mathcal{V}$ means that, for all $p \in (0, \infty), (pV_0, pV_p)$ is a pair of conjugated operators on $L^\infty(m)$ and so that, for all $p \in (0, \infty)$, we have the following relations:

$$V_0 = V_p(I + pV_0) \text{ and } V_p = V_0(I - pV_p.)$$

Throughout this section we shall consider $\mathcal{V} = (V_p)_{p\in(0,\infty)}$ a sub-Markov resolvent associated with the operator V_0 (on $L^\infty(m)$), while V_p is defined on $\mathcal{M}_l(m)$ with values in $\mathcal{M}_l(m)$ (for all $p \in [0, \infty)$).

Definition 3.3. *For all $p \in (0, \infty)$, we shall denote $L_p : \mathcal{M}_l(m) \to \mathcal{M}_l(m)$ the operator $L_p f := V_p(pf)$ and $\mathcal{S}_p := \mathcal{S}_{L_p}$.*

*All functions $u \in \bigcap_{p\in(0,\infty)} \mathcal{S}_p$ are called $\mathcal{V}$-**supermedian functions** and $\mathcal{S}_\mathcal{V} := \mathcal{S}(\mathcal{V}) := \bigcap_{p\in(0,\infty)} \mathcal{S}_p$.*

Remark 3.4. *(i) We have that: $u \in \mathcal{S}_\mathcal{V}$ if and only if, for all*

$$p \in (0, \infty), V_p(pu) \le u.$$

Moreover, for all $u \in \mathcal{S}_\mathcal{V}, V_0 0 \le u$ and for all $f \in \mathcal{M}_l(m)$ such that $f \ge 0$ we have that $V_0 f \in \mathcal{S}_\mathcal{V}$.

(ii). We shall define, for all $n \in \mathbb{N}, e_n := n + V_0 0$ and we shall have an increasing sequence $(e_n)_{n\in N}$ such that $e_n \in \mathcal{S}_\mathcal{V}$ (Remark 2.12 (ii)) and $\lim_{n\to\infty} e_n = \infty$, $m -$ a.e. on E.

(iii). By the definition $u \in \mathcal{M}_l(m)$ is $\mathcal{V}$-supermedian function if and only if for all $g \in L^\infty, g \le u$ we have that $V_p(pg) \le u$ and the last assertion is equivalent to the sentence: for all $n \in \mathbb{N}, V_p(p \inf(n, u)) \le u, \forall p \in (0, \infty)$.

Lemma 3.5. *We have the following assertions*

(i). For all $p, q \in (0, \infty)$ such that $p < q$ it will result that: (a). For all $u \in \mathcal{S}_q$, $V_p(pu) \le V_q(qu)$. (b). $\mathcal{S}_q \subset \mathcal{S}_p$.

(ii). For all $u \in \mathcal{S}_\mathcal{V}$, the function $(p \mapsto V_p(pu)) : (0, \infty) \to \mathcal{M}_l(m)$ is increasing.

(iii). Let $(u_n)_n \subset \mathcal{S}_\mathcal{V}$. (a). If $\inf_{n\in\mathbb{N}^\star} u_n \in \mathcal{M}_l(m)$, then $\inf_{n\in\mathbb{N}^\star} u_n \in \mathcal{S}_\mathcal{V}$. (b). If $(u_n)_n$ is increasing, then $\sup_{n\in\mathbb{N}^\star} u_n \in \mathcal{S}_\mathcal{V}$.

Proof. Obviously. □

Definition 3.6. *(i). Let $f \in \mathcal{M}_l(m)$. The map $inf\{u : u \in \mathcal{S}_{\mathcal{V}}$ and $u \geq f\}$ is called the* ***$\mathcal{S}_{\mathcal{V}}$-reduced function of f*** *and it is denoted by $R^{\upsilon} f := Rf$.*

(ii). It is obvious that, for all $p \in (0, \infty), f \leq R^{L_p} f \leq Rf, \forall f \in \mathcal{M}_l(m)$ (where L_p is the operator from Definition 3).

(iii). Moreover, for all $f \in \mathcal{M}_l(m)$ we have that $Rf = R(\sup(f, V_0 0))$.

Proposition 3.7. *For all $f \in \mathcal{M}_l(m)$ we have that $Rf \in \mathcal{S}_{\mathcal{V}}$.*

Proof. According to the Lemma 5,for all $p, q \in (0, \infty)$, $p < q$ we have that $R^{L_p} f \leq R^{L_q} f$ and by the Proposition 2.6 (i), $R^{L_p} f \in \mathcal{S}_p$.

Let $u := \sup\limits_{p\in(0,\infty)} R^{L_p} f$, since $R^{L_q} f \in \mathcal{S}_p$ provided that $p < q$ we have that

$$V_p(pR^{L_q} f) \leq R^{L_q} f \leq u,$$

and so that $V_p(pu) = \sup\limits_{q\in(0,\infty)} V_p(pR^{L_q} f) \leq \sup\limits_{q\in(0,\infty)} R^{L_q} f = u, \forall p \in (0, \infty)$. Therefore $u \in \mathcal{S}_{\mathcal{V}}$.

Obviously, $f \leq u$ and, if $s \in \mathcal{S}_{\mathcal{V}}$ is such that $f \leq s$, we shall have that $R^{L_p} f \leq s$ for all $p \in (0, \infty)$ and $u \leq s$. Therefore $u = Rf \in \mathcal{S}_{\mathcal{V}}$. □

Definition 3.8. *For each $f \in \mathcal{M}_l(m)$ we define*

$$V_\infty f := \sup_{n\in\mathbb{N}^\star} V_n(nf).$$

Remark 3.9. *Obviously $V_\infty f \in \mathcal{M}_l(m)$, $\forall f \in \mathcal{M}_l(m)$, so that*
$V_\infty : \mathcal{M}_l(m) \to \mathcal{M}_l(m)$.

Lemma 3.10. *The function V_∞ has the following properties:*

(i). V_∞ is increasing (hence V_∞ is a nonlinear operator on $\mathcal{M}_l(m)$).

(ii). V_∞ is continuous on the increasing sequences of $\mathcal{M}_l(m)$.

(iii). V_∞ is a sub-Markov operator from $\mathcal{M}_l(m)$ to $\mathcal{M}_l(m)$.

Proof. The assertions (i). and (ii). are obvious.

(iii). Let $f, g \in \mathcal{M}_l(m)$, and $\alpha \in (0, \infty)$ be such that $f \leq g + \alpha$. Since for each $n \in \mathbb{N}^\star$ L_n is a sub-Markov operator we have that:

$$V_n(nf) \leq V_n(ng) + \alpha \Rightarrow V_\infty f = \sup_{n\in\mathbb{N}^\star} V_n(nf) \leq \sup_{n\in\mathbb{N}^\star} V_n(ng) + \alpha = V_\infty g + \alpha.$$

□

Theorem 3.11. *We have the relation: $\mathcal{S}_{\mathcal{V}} = \mathcal{S}(V_\infty)$.*

Proof. Obviously that $\mathcal{S}_{\mathcal{V}} \subset \mathcal{S}(V_\infty)$.

If $s \in L^\infty(m) \cap \mathcal{S}(V_\infty)$, then $s \in \bigcap\limits_{n=1}^{\infty} \mathcal{S}_n$. In view of Lemma 3.5:

$$\forall\, n \in \mathbb{N}^\star,\ \forall\, p \in \mathbb{N}^\star,\ \forall\, p \in (0, n),\ \mathcal{S}_n \subset \mathcal{S}_p \Rightarrow s \in \bigcap_{p\in(0,\infty)} \mathcal{S}_p = \mathcal{S}_{\mathcal{V}}.$$

□

Corollary 3.12. *If follows that: $R^{V_\infty} = R^{\mathcal{V}}$.*

Proof. The assertion is obvious. □

Definition 3.13. *For all $f, g \in \mathcal{M}_l(m)$ we shall say that f is specific smaller than g will respect to $\mathcal{V}$ iff f is specific smaller than g with respect to V_∞. If $f, g \in \mathcal{M}_l(m)$ are as above then we shall denote this by $f \ll g$.*

Proposition 3.14 (a Mokobodzki type theorem). *For all $f \in \mathcal{M}_l(m)$ and $t \in \mathcal{S}_{\mathcal{V}}$. such that $f \ll t$, we shall have that $R^{\mathcal{V}} f \ll t$.*

Proof. Since $R^{\mathcal{V}} = R^{V_\infty}$ in view of Theorem 2.8 the assertion is obvious. □

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Department of Mathematics, University of Piteşti,
Str. Târgu din Vale, 110040 Piteşti, Romania
E-mail address: corneliu_udrea@yahoo.com