THE RELATION BETWEEN INCIDENCE COALGEBRA AND PATH COALGEBRA OF A PARTIAL ORDERED SET

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Abstract: *In the first part of the article are related some notion in a cathegoricaly way, like k-algebra and k-coalgebra, where k is a field. Then we construct the incidence coalgebra* (kS,Δ,ε) *and path coalgebra* $(kQ,\Delta',\varepsilon')$ *for a partial ordered set* $(P,\le)$*, and in the final part of the paper we find a reletion between them, more exactly an injective application* $f:kS\to kQ$.

1. Preliminary notions

Let k be a commutative field.

Definition. We say that a triplet (A,M,u) is a k - algebra, if A is a k – vector space, $M:A\otimes A\to A$ and $u:k\to A$ are linear applications of k –vector spaces such that $M\circ(M\otimes I)=M\circ(I\otimes M)$ and $M\circ(I\otimes u)=M\circ(u\otimes I)$.

Observation. The definition above is equivalent with the classical one, who claims A to be an unitary ring and to exist an application $\phi:k\to A$, with the condition $\operatorname{Im}\phi\subseteq Z(A)$. If we put $a\cdot b=M(a\otimes b)$, this multiplication give us an unitary ring structure on A, with unit $u(1)$.

Definiton. We say that a triplet (C,Δ,ε) is a k - coalgebra, if C is a k-vector space, $\Delta:C\to C\otimes C$ and $\varepsilon:C\to k$ are linear applications of k –vector spaces such that $(\Delta\otimes I)\circ\Delta=(I\otimes\Delta)\circ\Delta$ and $(\varepsilon\otimes I)\circ\Delta=(I\otimes\varepsilon)\circ\Delta$.

The application Δ is called the *comultiplication* and ε is called the *counit* of the coalgebra C.

2. Incidence coalgebra of a partial ordered set.

Let $(P,\le)$ be a partial ordered set (poset). We assume that the set P is local finite; it means that for every $x,y\in P$ such that $x\le y$, the set $[x,y]=\{z\in P|x\le z\le y\}$ is a finite one. Let $S=\{[x,y]|x,y\in P,x\le y\}$.

Let k be a commutative field and kS the vector space over k with the base S.

We obtain $kS=\left\{\sum_{i=1}^{n}a_{[x_i,y_i]}[x_i,y_i]\,\middle|\,[x_i,y_i]\in S, a_{[x_i,y_i]}\in k, n\in\mathrm{N}\right\}$. On this vector space we define a coalgebra structure:

$$\Delta:kS\to kS\otimes kS\ ,\ \Delta([x,y])=\sum_{x\le z\le y}[x,z]\otimes[z,y]$$

$$\varepsilon:kS\to k\ ,\ \varepsilon([x,y])=\delta_{x,y}\ .$$

(kS,Δ,ε) it is the *incidence coalgebra* of the partial ordered set P.

It is very simple to prove that Δ verifies $(\Delta\otimes I)\circ\Delta=(I\otimes\Delta)\circ\Delta$, and ε verifies $(I\otimes\varepsilon)\circ\Delta=(\varepsilon\otimes I)\circ\Delta$. We have:

$$(\Delta\otimes I)\circ\Delta([x,y])=(\Delta\otimes I)\left(\sum_{x\le z\le y}[x,z]\otimes[z,y]\right)=\sum_{x\le z\le y}\Delta([x,z])\otimes[z,y]=$$
$$=\sum_{x\le z\le y}\sum_{x\le t\le z}[x,t]\otimes[t,z]\otimes[z,y]$$
$$(I\otimes\Delta)\circ\Delta([x,y])=(I\otimes\Delta)\left(\sum_{x\le z\le y}[x,z]\otimes[z,y]\right)=\sum_{x\le z\le y}[x,z]\otimes\Delta([z,y])=$$
$$=\sum_{x\le z\le y}\sum_{z\le t\le y}[x,z]\otimes[z,t]\otimes[t,y],$$

from where we have $(\Delta\otimes I)\circ\Delta=(I\otimes\Delta)\circ\Delta$.

Also, $(I\otimes\varepsilon)\circ\Delta([x,y])=(I\otimes\varepsilon)\left(\sum_{x\le z\le y}[x,z]\otimes[z,y]\right)=\sum_{x\le z\le y}[x,z]\otimes\varepsilon([z,y])=$
$=\sum_{x\le z\le y}[x,z]\otimes\delta_{z,y}=[x,y]\otimes 1$ and

$$(\varepsilon\otimes I)\circ\Delta([x,y])=(\varepsilon\otimes I)\left(\sum_{x\le z\le y}[x,z]\otimes[z,y]\right)=\sum_{x\le z\le y}\varepsilon([x,z])\otimes[z,y]=$$
$$=\sum_{x\le z\le y}\delta_{x,z}\otimes[z,y]=1\otimes[x,y],$$

and because $[x,y]\otimes 1=1\otimes[x,y]$, we obtain $(I\otimes\varepsilon)\circ\Delta=(\varepsilon\otimes I)\circ\Delta$.

The kS^* - left module structure of kS is:

$$kS^*\times kS\to kS,\ \left(f,\sum_{i=1}^{n}a_{[x_i,y_i]}[x_i,y_i]\right)=\sum_{i=1}^{n}a_{[x_i,y_i]}\sum_{x_i\le z\le y_i}f([z,y_i])[x_i,z],\text{ for}$$
$$\Delta([x_i,y_i])=\sum_{x_i\le z\le y_i}[x_i,z]\otimes[z,y_i].$$

The kS^* - right module structure of kS is:

$$kS\times kS^*\to kS,\ \left(\sum_{i=1}^{n}a_{[x_i,y_i]}[x_i,y_i],f\right)=\sum_{i=1}^{n}a_{[x_i,y_i]}\sum_{x_i\le z\le y_i}f([x_i,z])[z,y_i]\text{ for}$$
$$\Delta([x_i,y_i])=\sum_{x_i\le z\le y_i}[x_i,z]\otimes[z,y_i].$$

3. Path coalgebra

A *quiver* is a pair $Q=(Q_0,Q_1)$, where Q_0 is the set of vertices and Q_1 is the set of the arrows between vertices. Let $s:Q_1\to Q_0$ and $t:Q_1\to Q_0$ be two applications, where $s(\alpha)=i$ and $t(\alpha)=j$, for every arrow $\alpha:i\to j$ from the vertex i to the vertex j.

We call a *path* in the quiver Q a sequence $p=\alpha_n...\alpha_1$, with $t(\alpha_i)=s(\alpha_{i+1}), i=1,...,n$. A trivial path, noted with e_i, is a path with the property $t(e_i)=s(e_i)=i$. For a nontrivial path $p=\alpha_n...\alpha_1$ we put $s(p)=s(\alpha_1)$ and $t(p)=t(\alpha_n)$. A path p is called *cycle* if $s(p)=t(p)$. The length of a path p is $|p|$.

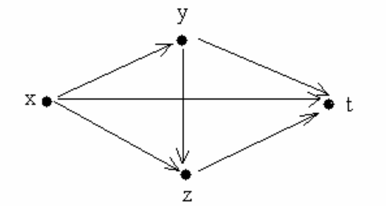
Now, let be $(P,\le)$ a poset locally finite and (kS,Δ,ε) its incidence coalgebra.

We can construct the oriented quiver $Q=(Q_0,Q_1)$ in the following way:

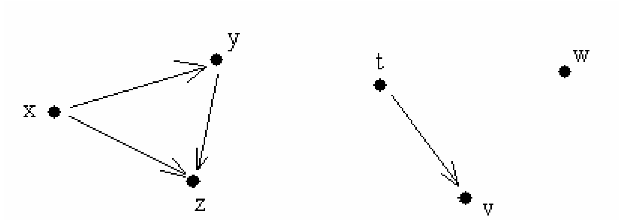
- $Q_0 = P$, and for every $x, y \in P$ we put $\alpha : x \to y$, $\alpha(x) = \begin{cases} x \to y, & if\ x \le y \\ 0, & else \end{cases}$; it means that there exist an arrow from x to y if and only if $x \le y$;

- Q_1 is the set of all the arrows between the vertices from Q_0.

Example 1. If $P = \{x, y, z, t\}$ with $x \le y \le z \le t$, the quiver $Q = (Q_0, Q_1)$ is:



Exemple 2. If $P = \{x, y, z, t, v, w\}$ is a poset such that: $x \le y \le z$ and $t \le v$, the quiver $Q = (Q_0, Q_1)$ is:



Proposition. The quiver $Q = (Q_0, Q_1)$ associated to the poset $(P, \le)$ has no oriented cycles .

Let k be a field, $(P, \le)$ a poset and $Q = (Q_0, Q_1)$ the quiver associated to P. Now we construct a k – vector space kQ with base Q. We obtain that $kQ = \left\{ \sum_{i=1}^{n} a_i p_i \middle| a_i \in k, p_i\ drum\ in\ Q, n \in N^* \right\}$. Let's define a coalgebra structure:

$$\Delta' : kQ \to kQ \otimes kQ,\ \Delta'(p) = \sum_{p = p_1 p_2} p_1 \otimes p_2$$

$$\varepsilon' : kQ \to k,\ \varepsilon'(p) = \delta_{|p|,0}$$

where for a path $p_2 = \alpha_s ... \alpha_1$, $1 \le s \le t$ we know that $|p| = t$ is it's lenght.

The triplet $(kQ, \Delta', \varepsilon')$ is called the *path coalgebra* of $(P, \le)$.

We observe that for every path $p \in \mathbf{P}$ of finite length, the number of pairs (p_1, p_2) with $p = p_1 p_2$ is a finite one, and so, the sum which appears in $\Delta'(p)$ is finite.

It is obvious that $(kQ, \Delta', \varepsilon')$ is indeed coalgebra. We have:

$$(I \otimes \Delta')\Delta'(p) = (I \otimes \Delta')\left(\sum_{p = p_1 p_2} p_1 \otimes p_2 \right) = \sum_{p = p_1 p_2} p_1 \otimes \Delta'(p_2) =$$

$$= \sum_{p = p_1 p_2} \sum_{p_2 = p_{21} p_{22}} p_1 \otimes p_{21} \otimes p_{22} = \sum_{p = p_1 p_2 p_3} p_1 \otimes p_2 \otimes p_3 \text{ and}$$

$$(\Delta' \otimes I)\Delta'(p) = (\Delta' \otimes I)\left(\sum_{p = p_1 p_2} p_1 \otimes p_2 \right) = \sum_{p = p_1 p_2} \Delta'(p_1) \otimes p_2 =$$

$$= \sum_{p = p_1 p_2} \sum_{p_1 = p_{11} p_{12}} p_{11} \otimes p_{12} \otimes p_2 = \sum_{p = p_1 p_2 p_3} p_1 \otimes p_2 \otimes p_3,$$

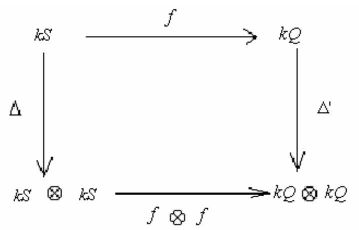
and so $(I\otimes\Delta')\Delta'(p) = (\Delta'\otimes I)\Delta'(p)$, for every path p in Q.

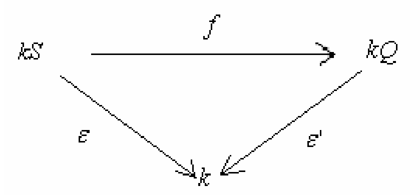
More, we have, $(\varepsilon'\otimes I)\Delta(p)=(\varepsilon'\otimes I)\left(\sum_{p=p_1p_2} p_1\otimes p_2\right)=\sum_{p=p_1p_2}\varepsilon'(p_1)\otimes p_2 = 1\otimes p$ and $(I\otimes\varepsilon')\Delta(p)=(I\otimes\varepsilon')\left(\sum_{p=p_1p_2} p_1\otimes p_2\right)=\sum_{p=p_1p_2} p_1\otimes\varepsilon'(p_2)=p\otimes 1$, so $(\varepsilon'\otimes I)\Delta(p)$ = $(I\otimes\varepsilon')\Delta(p)$, for every path p in Q.

4. The relation between incidence coalgebra and path coalgebra of a poset

Theorem. Let $(P,\le)$ be a poset, $Q=(Q_0,Q_1)$ the quiver associated to $(P,\le)$, $(kQ,\Delta',\varepsilon')$ the path coalgebra and (kS,Δ,ε) the incidence coalgebra. Then does exist an injectif application of $k-$coalgebras $f:kS\to kQ$.

Prof. Let $f:kS\to kQ$, $f\left(\sum_{i=1}^{n} a_{[x_i,y_i]}[x_i,y_i]\right)=\sum_{i=1}^{n} a_{[x_i,y_i]}\sum p_i$, where p_i is a path from x_i to y_i. From the definition of f it is obvious that f is a liniar application. Now, let prove that the two next diagrams are commutative (means that f is a $k-$morpfism of coalgebras).





Because S is a base for the coalgebra kS, it is enough to prove the commutability of the first diagram for every interval $[x,y]\in S$. And so we have:

$(\Delta'\circ f)([x,y])=\Delta'(f([x,y]))=\Delta'\left(\sum p\right)=\sum\Delta'(p)=\sum_{p}\sum_{p=p_2p_1} p_2\otimes p_1$ and

$$((f\otimes f)\circ\Delta)([x,y])=(f\otimes f)(\Delta([x,y]))=(f\otimes f)\left(\sum_{x\le z\le y}[x,z]\otimes[z,y]\right)=$$
$$=\sum_{x\le z\le y} f([x,z])\otimes f([z,y])=\sum_{x\le z\le y}\left(\sum q_1\right)\otimes\left(\sum q_2\right),$$

where p is a path from x to y, q_1 is a path from x to z and q_2 is a path from z to y.

But $x\le z\le y$ if and only if does exist an arrow from x to z and another one from z to y, that implies $\sum_{p}\sum_{p=p_2p_1} p_2\otimes p_1=\sum_{x\le z\le y}\left(\sum q_1\right)\otimes\left(\sum q_2\right)$ and so $(\Delta'\circ f)([x,y])=((f\otimes f)\circ\Delta)([x,y])$, for every $[x,y]\in S$.

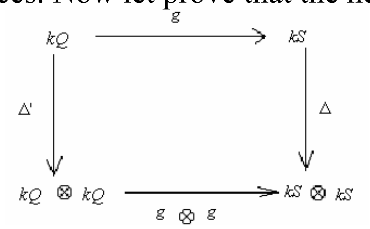
In the same way we prove $(\varepsilon'\circ f)([x,y])=\varepsilon'(f([x,y]))=\varepsilon'\left(\sum p\right)=$ $=\sum\varepsilon'(p)=\sum\delta_{|p|,0}=\begin{cases}1\ ,\ if\ \ x=y\\ 0\ ,\ else\end{cases}$, where p is a path from x to y, and $\varepsilon([x,y])=\delta_{x,y}=\begin{cases}1\ ,\ if\ \ x=y\\ 0\ ,\ else\end{cases}$. It is clear that $(\varepsilon'\circ f)([x,y])=\varepsilon([x,y])$, for every $[x,y]\in S$.

Now let prove that f is injective. Because S is a base for the coalgebra kS it is enough to prove that if $f([x,y])=f([z,t]),\forall[x,y],[z,t]\in S$, then $[x,y]=[z,t]$.

$f([x,y]) = f([z,t])$ implies $\sum_p p = \sum_q q$, where p is a path from x to y, and q is a path from z to t. We obtain that the paths from x to y are the same with those from z to t, and then $x = z$ and $y = t$, means that $[x,y] = [z,t]$.

Observation. Let $(P,\le)$ be a poset, $Q = (Q_0, Q_1)$ the quiver we associate to $(P,\le)$, $(kQ, \Delta', \varepsilon')$ path coalgebra and $(kS, \Delta, \varepsilon)$ incidence coalgebra. Then the application $g : kQ \to kS$, $g\left(\sum_{i=1}^{n} a_i p_i\right) = \sum_{i=1}^{n} a_i [x_1^i, x_{t+1}^i]$, where if $p_i = \alpha_t^i ... \alpha_1^i$, $s(\alpha_1^i) = x_1^i$ and $t(\alpha_t^i) = x_{t+1}^i$, is not a coalgebra morpfism.

Prof. The application g is well definited because if $p_i = \alpha_t^i ... \alpha_1^i$ with $\alpha_j^i : x_j^i \to x_{j+1}^i$, for every $j \in \{1,...,t\}$, then $x_j^i \le x_{j+1}^i$, for every $j \in \{1,...,t\}$. And so we have $x_1^i \le ... \le x_{t+1}^i$, means that the interval $[x_1^i, x_{t+1}^i] \in S$ does exist. From the expression of g, it is obvious that it is a $k-$application of vector spaces. Now let prove that the next diagram is not commutative:



Because Q is a base for the coalgebra kQ it is enough to prove that the diagrame above is not commutative for some path $p \in Q$. Let $p = \alpha_t ... \alpha_1$, with $\alpha_i : x_i \to x_{i+1}$, for every $i \in \{1,...,t\}$.

We have $(\Delta \circ g)(p) = \Delta([x_1, x_{t+1}]) = \sum_{x_1 \le z \le x_{t+1}} [x_1, z] \otimes [z, x_{t+1}]$ and

$$((g \otimes g) \circ \Delta)(p) = (g \otimes g)\left(\sum_{p = p_2 p_1} p_2 \otimes p_1\right) = (g \otimes g)\left(\sum_{1 < s \le t_1} \alpha_t ... \alpha_s \otimes \alpha_{s-1} ... \alpha_1\right) =$$

$$= \sum_{1 < x \le t+1} g(\alpha_t ... \alpha_s) \otimes g(\alpha_{s-1} ... \alpha_1) = \sum_{1 < s \le t+1} [x_s, x_{t+1}] \otimes [x_1, x_s],$$

and the prof is ended.

Now, let $Q = (Q_0, Q_1)$ be a quiver. We put $P = Q_0$, where Q_0 is the set of vertices from Q. We define on P a relation $\le$ in the following way: for $x, y \in P$ we say that $x \le y$ if and only if does exist $\alpha : x \to y$ arrow from x to y in the quiver Q. We put the condition that P be partial ordered (i.e. $\le$ is reflexive, antisimetric and transitive).

$(P,\le)$ is reflexive if for every $x \in P$ we have $x \le x$, means that exist an arrow from x to x in Q, which is obvious.

$(P,\le)$ is antisimetric if for every $x, y \in P$, from $x \le y$ and $y \le x$ results that $x = y$, means that if does exist an arrow from x to y and another from y to x in Q, then $x = y$. This is possible only if Q does not have oriented cycles.

$(P,\le)$ is transitive if for every $x, y, z \in P$ such that $x \le y$ and $y \le z$ we have $x \le z$, means that if does exist arrow from x to y and another from y to z in Q, then we have arrow from x to z.

If $Q=(Q_0,Q_1)$ verify all this conditions, then $(P,\leq)$ is a poset and we can define a k – incidence coalgebra (kS,Δ,ε), and a k – path coalgebra $(kQ,\Delta',\varepsilon')$.

If we note with R the set of all finite quivers with the properties above and P the set of all locat finite posets $(P,\leq)$, it is obvious that these two sets are equivalent.

Let C be the set of all k – coalgebras (kS,Δ,ε), where S is the set of all intervals of a poset $P\in P$. We note with C' the set of all k – coalgebras $(kQ,\Delta',\varepsilon')$, where $Q\in$ R . Then C and C' are equivalent categories, and all the properties of an incidence coalgebra can be studied through those of the path coalgebra and reverse.

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