WAYS TO OBTAIN THE BERNSTEIN-STANCU OPERATORS

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Abstract: *In this paper we present a probabilistic way to obtain Bernstein Stancu type operators.*

1. In [1] D.D. Stancu defined for two positive numbers $0 \le \alpha \le \beta$ independent of n and for any function $f \in C[0,1]$ the operator :

$$(P_n^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right). \tag{1}$$

The Bernstein-Stancu operator uses the equidistant nodes $a_0 = \frac{\alpha}{n+\beta}$, $a_1 = x_0 + h$, …, $a_n = x_0 + nh$ where $h = \frac{1}{n+\beta}$ and because

$$\left(P_n^{(\alpha,\beta)} f\right)(0) = f\left(\frac{\alpha}{n+\beta}\right)$$

$$\left(P_n^{(\alpha,\beta)} f\right)(1) = f\left(\frac{n+\alpha}{n+\beta}\right),$$

Bernstein-Stancu operator interpolates function f in $x=0$ if $\alpha = 0$ and in $x=1$ if $\alpha = \beta$.

Values on test function are given by:

$$\left(P_n^{(\alpha,\beta)} e_0\right)(x) = 1 \tag{2}$$

$$\left(P_n^{(\alpha,\beta)} e_1\right)(x) = x + \frac{\alpha - \beta x}{n+\beta} \tag{3}$$

$$\left(P_n^{(\alpha,\beta)} e_2\right)(x) = x^2 + \frac{nx(1-x) + (\alpha - \beta x)(2nx + \beta x + \alpha)}{(n+\beta)^2} \tag{4}$$

so we can state that for any $f \in C[0,1]$ the sequence $\left((P_n^{(\alpha,\beta)} f)(x)\right)_{n \in \mathbb{N}}$ converges uniformly to $f(x)$ on [0,1].

2. Another way to construct this operator is presented in [3] where it is considered a function f, bounded on [0,1], the positive numbers $0 \le \alpha \le \beta$ and for $n \in \mathbb{N}$ and $x \in 0,1]$ the poligonal lines $g_{px,n}$ are defined:

$$\left(\frac{px+k+\alpha}{n+\beta}, g_{(p-1)x,n}\left(\frac{px+k+\alpha}{n+\beta}\right)\right) \qquad j = \overline{0, n-p}, \qquad p = \overline{1,n}$$

with $g_{0,n}$ given by:

$$\left(\frac{j+\alpha}{n+\beta}, f\left(\frac{j+\alpha}{n+\beta}\right)\right) \qquad j=\overline{0,n}.$$

Using the Lagrange interpolation formula

$$L(f;a;b;t)=\frac{t-b}{a-b}f(a)+\frac{t-a}{b-a}f(b)$$

the popligonal lines can be written as

$$g_{px,n}(t)=\left[1-(n+\beta)t+px+j+\alpha\right]g_{(p-1)x,n}\left(\frac{px+j+\alpha}{n+\beta}\right)+$$
$$+\left[(n+\beta)t-px-j-\alpha\right]g_{(p-1)x,n}\left(\frac{px+j+1+\alpha}{n+\beta}\right),$$

for any

$$t\in\left[\frac{px+j+\alpha}{n+\beta},\frac{px+j+1+\alpha}{n+\beta}\right], \qquad j=\overline{0,n-p-1}, \qquad p=\overline{0,n-1}.$$

From the construction of poligonal lines we have

$$g_{px,n}\left(\frac{px+j+\alpha}{n+\beta}\right)=g_{(p-1)x,n}\left(\frac{px+j+\alpha}{n+\beta}\right)$$
$$g_{px,n}\left(\frac{px+j+1+\alpha}{n+\beta}\right)=g_{(p-1)x,n}\left(\frac{px+j+1+\alpha}{n+\beta}\right)$$

therefore

$$g_{px,n}\left(\frac{px+j+\alpha}{n+\beta}\right)=\sum_{k=0}^{p}\binom{p}{k}x^k(1-x)^{p-k}f\left(\frac{k+j+\alpha}{n+\beta}\right).$$

For $p=n$ and $j=0$ we obtain the Stancu operator $\left(P_n^{(\alpha,\beta)}f\right)(x)$.

3. Now we consider the Newton intepolation polynomial of a function f on nodes $a_k=\frac{k+\alpha}{n+\beta}, k=0,1,\ldots,n$ with step $h=\frac{1}{n+\beta}$; we have:

$$(N_nf)(x)=f(a_0)+\sum_{k=0}^{n}(x-a_0)\ldots(x-a_{k-1})[a_0,a_1,\ldots,a_k;f]=$$
$$=f(a_0)+\sum_{k=0}^{n}\left(x-\frac{\alpha}{n+\beta}\right)\ldots\left(x-\frac{\alpha+k-1}{n+\beta}\right)[a_0,a_1,\ldots,a_k;f]=$$
$$=f(a_0)+\sum_{k=0}^{n}\left((n+\beta)x-\alpha\right)\ldots\left((n+\beta)x-(\alpha+k-1)\right)\frac{1}{(n+\beta)^k}[a_0,a_1,\ldots,a_k;f]=$$
$$=f(a_0)+\sum_{k=0}^{n}\left((n+\beta)x-\alpha\right)\ldots\left((n+\beta)x-(\alpha+k-1)\right)\frac{\Delta_h^k f(a_0)}{k!}=$$
$$=f(a_0)+\sum_{k=0}^{n}\left((n+\beta)x-\alpha\right)^{[k]}\frac{\Delta_h^k f(a_0)}{k!}.$$

If we denote $(n+\beta)x-\alpha=y$ then $x=\frac{y+\alpha}{n+\beta}$ and we get

$$(N_n f)\left(\frac{y+\alpha}{n+\beta}\right) = f(a_0) + \sum_{k=0}^{n} y^{[k]} \frac{\Delta_h^k f(a_0)}{k!}. \tag{5}$$

Now we consider a random variable Y defined as:

$$Y : \begin{pmatrix} k \\ p_{nk} \end{pmatrix}, \quad p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0,1,\ldots,n, \quad x \in 0,1].$$

The mean value of the random variable $(N_n f)\left(\frac{Y+\alpha}{n+\beta}\right)$ is given by

$$E\left((N_n f)\left(\frac{Y+\alpha}{n+\beta}\right)\right) = \sum_{k=0}^{n}\left(f(a_0) + \sum_{i=0}^{n} k^{[i]} \frac{\Delta_h^i f(a_0)}{i!}\right) p_{nk}(x) =$$

$$= f(a_0) + \sum_{i=1}^{n} \frac{\Delta_h^i f(a_0)}{i!} \sum_{k=0}^{n} k^{[i]} p_{nk}(x).$$

If we take $g(t) = E(t^Y)$ then

$$g(t) = \sum_{k=0}^{n} t^k p_{nk}(x) = (1-x+tx)^n,$$

and making $t = 1$ in the expression of derivative of order $i \geq 0$

$$g^{(i)}(t) = \sum_{k=0}^{n} k^{[i]} t^{k-i} p_{nk}(x) = n^{[i]} x^i (1-x+tx)^{n-i}$$

we obtain

$$g^{(i)}(1) = \sum_{k=0}^{n} k^{[i]} p_{nk}(x) = n^{[i]} x^i.$$

so the mean value can be written as:

$$E\left((N_n f)\left(\frac{y+\alpha}{n+\beta}\right)\right) = f(a_0) + \sum_{i=1}^{n} \frac{\Delta_h^i f(a_0)}{i!} n^{[i]} x^i =$$

$$= f(a_0) + \sum_{i=1}^{n} \binom{n}{i} \Delta_h^i f(a_0) x^i.$$

On the other hand the Newton polynomial is interpolatory on nodes a_k, therefore

$$E\left((N_n f)\left(\frac{Y+\alpha}{n+\beta}\right)\right) = \sum_{k=0}^{n} (N_n f)\left(\frac{k+\alpha}{n+\beta}\right) p_{nk}(x) =$$

$$= \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) p_{nk}(x) = \left(P_n^{(\alpha,\beta)} f\right)(x),$$

and it follows that

$$\left(P_n^{(\alpha,\beta)} f\right)(x) = f(a_0) + \sum_{i=1}^{n} \binom{n}{i} \Delta_h^i f(a_0) x^i.$$

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SOME REASONS TO FUZZY APPROACH OF THE CHOICE FUNCTIONS

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Abstract. *The human preferences and the choice represent a significant problem in many domains as the decision theory, economics or social life. In the real life there are a many choice function that are not rationalizable. The specialized literature gives as procedures which imbedded the non-rational functions in to one rational. A full of advantages method that treats the non-rational choice functions is the utilization to fuzzy theory in the choice problems.*

1. **Introduction.** A choice function are designed for describe a choice behaviour and it selects an object from a finite set X= $\{x_1,...x_I\}$ of I objects.

Definition: Let P(X) a collection of A, B,… nonempty subsets of X. A single-valued **choice function c** on P(X) is

$$c: P(X) \rightarrow X$$
$$\text{with } c(A) \in A \text{ for every } A \in P(X)$$

Definition: For previous function, a **preference relation** $\succ$ is said to **rationalize c** if and only if

$$c(A)=x,\ x \in A \text{ and } x \succ y \text{ for every } y \in A,\ y \neq x$$

In these conditions the function c is named **rational choice function.**
The rational choice functions have the following property (see [1]):

Property: If A, B $\in$ P(X), $A \subseteq B$ and c(B) $\in$ A, then c(A)=c(B).

Also,
A choice function c is rationalizable $\Leftrightarrow$ c satisfies the previous property.

Observations:
A binary relation on X, $\succ$ is preference relation if is irreflexive, transitive and total. A preference relation rationalizes a choice function c when it chooses the most preferred object from a set A.

But in the real life there are a many choice function that are not rationalizable.

2. Reasons for fuzzy approach.