APPLICATIONS OF Itô STOCHASTICALLY INTEGRAL IN THE EVALUATION OF THE EXPECTATION VALUES OF BROWNIAN MOTION

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Abstract: *In the paper below are evaluated the expectation values of the natural powers of Brownian motion. It is given the definition of these evaluations and will be demonstrated the recurrent relation between the terms of the same row. In these evaluations It is used the integral stochastic Itô calculation.*

1. Introduction

The Brownian movement modulates systems with a large amount of random and independent effects, which if considered apart from the others they have no effect on the system, but which in total, gathered, these effects generate a stochastic Gaussian process. According to Kolmogorov theory, it is enough to generate the Brownian motion $\{B_t\}_{t\geq 0}$, naming a family $\{v_{t_1},...,v_{t_k}\}$ of probability values that meet the conditions of this theory.

***Summary*:** According to the things mentioned above we remind the definition of Brownian motion.

***Definition* 1:** The process $\{B_t\}_{t\geq 0}$ with a continuum parameter, it is a Brownian motion if :([7],pag.117)

1. The increases $B_{t+\tau} - B_t$, $\tau > 0$ are independent and stationary
2. trajectories $\{B_t\}_{t\geq 0}$ are continuous
3. for any $t \geq 0$, B_t it has a normal repartition
4. $E[B_t] = 0$, $E[B_t^2] = t$, for any $t \geq 0$

Theorem: Let be a one-dimensional Brownian motion $\{B_{t\geq 0}\} \subset R$, $B_0 = 0$.we define the row of Brownian motion's expectation values of natural powers:

$$\{\beta_n(t)\}_{n\in N},\ \beta_n(t) = E[B_t^n],\ \beta_0(t) = 1,\ \beta_1(t) = 0 \tag{1}$$

Then the following relation of recurrent becomes true for n a natural bigger or equal to 2:

$$\beta_n(t) = \frac{1}{2}n(n-1)\int_0^t \beta_{n-2}(s)ds \tag{2}$$

More than that, if n par and n impair we have:

$$\beta_{2k}(t)=\frac{(2k)!}{2^k k!}t^k,\ k\in N^*,\ \beta_{2k+1}(t)=0. \tag{3}$$

Demonstration: We start from the stochastic Itô integral, ([9], relation (4)):

$$\int_0^t B_s^n dB_s = \frac{1}{n+1}B_s^{n+1}\Big|_0^t - \frac{n}{2}\int_0^t B_s^{n-1}ds \tag{4}$$

We obtain:

$$\frac{1}{n+1}B_s^{n+1}\Big|_0^t = \int_0^t B_s^n dB_s + \frac{n}{2}\int_0^t B_s^{n-1}ds \tag{4'}$$

$$B_s^{n+1}\Big|_0^t = (n+1)\int_0^t B_s^n dB_s + \frac{n(n+1)}{2}\int_0^t B_s^{n-1}ds \tag{4''}$$

We rewrite the relation for n:

$$B_s^{n}\Big|_0^t = n\int_0^t B_s^{n-1} dB_s + \frac{n(n-1)}{2}\int_0^t B_s^{n-2}ds \tag{4'''}$$

As $B_0=0$ and using the expectation's properties from [7] (pag. 17) we obtain from (4''')

$$E\left[B_t^n\right] = nE\left[\int_0^t B_s^{n-1}dB_s\right] + \frac{n(n-1)}{2}E\left[\int_0^t B_s^{n-2}ds\right] \tag{5}$$

As $E\left[\int_0^t f(s)dB_s\right]=0$, [11], (page 30); statements (2) it is demonstrated. For particularity values of $n\in N$, it is easy to verify:

$$n=0,\ \beta_0(t)=E\left[B_t^0\right]=E[1]=1$$

$$n=1,\ \beta_1(t)=E\left[B_t\right]=0,$$

$$n=2,\ \beta_2(t)=\frac{2\cdot 1}{2}\int_0^t \beta_0(s)ds=t$$

$$n=3,\ \beta_3(t)=\frac{3\cdot 2}{2}\int_0^t \beta_1(s)ds=0$$

$$n=4,\ \beta_4(t)=\frac{4\cdot 3}{2}\int_0^t \beta_2(s)ds=\frac{4!}{2^2\cdot 2!}t^2$$

After this we demonstrate the statement (3) by induction considering n. We state that

$$\beta_{2k+1}(t)=0,\ \beta_{2k}(t)=\frac{(2k)!}{2^k k!}t^k,\ k\in N$$

Using (2) we obtain:

$$\beta_{2(k+1)}(t)=\frac{2(k+1)(2k+1)}{2}\int_0^t \beta_{2k}(s)ds=(k+1)(2k+1)\frac{(2k)!}{2^k\cdot k!}\cdot\frac{t^{k+1}}{k+1}=\frac{(2k+2)!}{2^{k+1}\cdot(k+1)!}t^{k+1}$$

$$\beta_{2k+3}(t) = \frac{(2k+3)(2k+2)}{2}\int_0^t \beta_{2k+1}(s)ds = 0 .$$

This shows that statements (3) are true for any natural number n.

2. Conclusions

The row of one-dimensional Brownian movement's natural powers' expectations values is formed of two lesser rows: The lesser row of impair rank that is constantly zero, and the one of par rank which depends in time:

$$\beta_1(t) = E[B_t] = 0,\ \ \beta_3(t) = E[B_t^3] = 0 ,\ldots, \beta_{2k+1}(t) = E[B_t^{2k+1}] = 0,\ldots$$
$$\beta_0(t) = E[B_t^0] = 1,\ \ \beta_2(t) = E[B_t^2] = t ,\ \beta_4(t) = E[B_t^4] = 3t^2 ,\ \beta_6(t) = E[B_t^6] = 15t^3 ,\ldots$$

$$\beta_{2k}(t) = E[B_n^{2k}] = \frac{(2k)!}{2^k k!}t^k ,\ \ k \in N .$$

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