So, $\overrightarrow{\mathrm{KC}} = \overrightarrow{\mathrm{AA'}}$ equivalent to KC || AA'

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THE VECTOR ε ACCELERATION ALGORITHM

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Abstract: *The purpose of this article is to make a short introduction to the vector ε acceleration algorithm and give an example of how it can be used to approximate the solution of a linear system of equations.*

1. Section 1.

In this section is given the formula of the vector ε algorithm and a theorem related to the application of this algorithm to a sequence which satisfies a linear recursive equation.

Definition 1. *For any* $\mathbf{a} \in \mathbf{R}^p$, $\{0\}$ *we will denote by* $\mathbf{a}^{-1}$ *the following expression*

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{< \mathbf{a}, \mathbf{a} >}$$

It is easy to prove that the above definition for the inverse of a vector satisfies the following properties.

Proposition 2. *For any* $\mathbf{a} \in \mathbf{R}^p$, $\{0\}$ *we have*

1. $(\mathbf{a}^{-1})^{-1} = \mathbf{a}$
2. $< \mathbf{a}, \mathbf{a}^{-1} > = 1$

Definition 3. *Now we consider a sequence* $\{\mathbf{x}_n\}_{n \in N}$ *of vectors in* R^p *and define a double indexed sequence* $\varepsilon_k^{(n)}$ *by*

$$(1) \quad \begin{cases} \varepsilon_{-1}^{(n)} = 0_{R^p}, \varepsilon_0^{(n)} = x_n, n \in N \\ \varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + (\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)})^{-1}, n, k \in N \end{cases}$$

Equations 1 define the vector ε algorithm and they transform the initial sequence into a matrix with an infinite number of columns, each one being a sequence of vectors in R^p.

Theorem 4. *If we apply the vector ε algorithm to the sequence $\{\mathbf{x}_n\}_{n\in N} \subset R^p$ satisfying the equation*

$$(2) \quad \sum_{i=0}^{k} a_i(\mathbf{x}_{n+i} - \mathbf{x}) = 0, \forall n \geq n_0$$

where $a_i \in R$, then

$$\varepsilon_{2k}^{(n)} = \mathbf{x}, \forall n \geq n_0 \forall k \geq 0 \text{ if } \sum_{i=0}^{k} a_i \neq 0$$

and

$$\varepsilon_{2k}^{(n)} = 0, \forall n \geq n_0 \forall k \geq 0 \text{ if } \sum_{i=0}^{k} a_i = 0$$

We now recall the definition for the polynomial annihilator of a square matrix and associated vector.

Definition 5. *Given a square matrix $A \in M_p(R)$, a nonzero vector $\mathbf{x} \in R^p$ and an algebraic polynomial $P \in R[X]$, $\{0\}$ we say that P is an annihilator polynomial of the matrix A corresponding to the vector $\mathbf{x}$ if it satisfy the relation*

$$P(A) \cdot \mathbf{x} = 0_{R^p}$$

Obviously, if P is an annihilator polynomial then $a \cdot P$ is also an annihilator polynomial for the same matrix and vector. Therefore we may consider that the annihilator polynomial has the dominant coefficient equal to 1. The polynomial of minimal degree which satisfies the above definition is called annihilator minimal polynomial.

From Cayley–Hamilton theorem is easy to see that the characteristic polynomial $P(\lambda) = det(\lambda I - A)$ is an annihilator polynomial for the matrix A and any nonzero vector x. Moreover, the annihilator minimal polynomial divides the characteristic polynomial.

2. Section 2. Application of the vector ε algorithm to the solution of linear system of equations

Many iterative methods for approximating the solution of a linear system use the fixed point iteration. An usual approach is to generate a sequence of vectors by the following formula

$$\begin{cases} \mathbf{x}_0 \in R^p \\ \mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{b} \end{cases} \tag{3}$$

where $A \in M_p(R)$ and $\mathbf{b} \in R^p$. In case of convergence the limit x^* of the sequence $\{\mathbf{x}_n\}_{n\in N}$ is the solution of the linear system

$$(I - A)\mathbf{x} = \mathbf{b}$$

.

We have the following theorem.

Theorem 1. *If A and $I - A$ are non-singular matrices then applying the vector ε algorithm to the sequence $\{\mathbf{x}_n\}_{n\in N}$ generated by equation 3 we have*

$$\varepsilon_{2m}^{(n)} = \mathbf{x}^*, \forall n \geq 0$$

where $\mathbf{x}* = (I - A)^{-1}\mathbf{b}$ and m is the degree of the annihilator minimal polynomial of the matrix A and the corresponding vector $\mathbf{x}_0 - \mathbf{x}^*$.

Theorem 1 says that applying vector ε algorithm to the sequence $\{\mathbf{x}_n\}_{n\in N}$ we obtain in the column $2m$ of the ε table the exact solution of the linear system $(I-A)\mathbf{x}=\mathbf{b}$ whatever the convergence of the initial sequence.
Since the annihilator minimal polynomial divides the characteristic polynomial we have $m \le p$. This fact tells us that we can get the exact solution computing at most $2p$ columns in the ε table.

Example Let us consider the following system of equations

$$(4)\ \begin{cases} 2a+b+c=2 \\ a+2b+c=0 \\ a+b+2c=2 \end{cases}$$

With exact solution (a,b,c)=(1,-1,1). The well known Jacobi iteration produces a divergent vector sequence: $x_0=(0,0,0)$, $x_1=(4,0,4)$, $x_2=(-16,-16,-16)$, $x_3=(116, 112, 116)$, $x_4=(-800\ \text{-}800\ \text{-}800)$, $x_5=(5604, 5600, 5604)$, $x_6=(-39216, \text{-}39216, \text{-}39216)$, $x_7=(274516, 274512, 274516)$, $x_8=(-1921600, \text{-}1921600, \text{-}1921600)$, $x_9=(13451204, 13451200, 13451204)$

The second column in ε table is given by:

$\varepsilon_2^{(0)}$	0.54545454545454	−0.36363636363636	0.54545454545454
$\varepsilon_2^{(1)}$	1.38814531548757	−1.73613766730402	1.38814531548757
$\varepsilon_2^{(2)}$	0.62309164031079	−0.25192300183517	0.62309164031079
$\varepsilon_2^{(3)}$	1.37527322393441	−1.74972647724235	1.37527322393441
$\varepsilon_2^{(4)}$	0.62496095580173	−0.25003905029644	0.62496095580173
$\varepsilon_2^{(5)}$	1.37500557798921	−1.74999442188710	1.37500557798921
$\varepsilon_2^{(6)}$	0.62499920313712	−0.25000079686288	0.62499920313712
$\varepsilon_2^{(7)}$	1.37500011362135	−1.74999988637865	1.37500011362135

The fourth column in ε table is given by:

$\varepsilon_4^{(0)}$	0.99999999999999	−1.00000000000001	0.99999999999999
$\varepsilon_4^{(0)}$	0.99999999999988	−1.00000000000009	0.99999999999988
$\varepsilon_4^{(0)}$	0.99999999999964	−1.00000000000024	0.99999999999964
$\varepsilon_4^{(0)}$	0.99999999999904	−1.00000000000098	0.99999999999904
$\varepsilon_4^{(0)}$	0.99999999999826	−1.00000000000142	0.99999999999826
$\varepsilon_4^{(0)}$	0.99999999994449	−1.00000000005445	0.99999999994449

If we use now the Gauss-Seidel iteration we obtain the below convergent sequence

x_0	0	0	0
x_1	1.00000000000000	−0.50000000000000	0.75000000000000
x_2	0.87500000000000	−0.81250000000000	0.96875000000000
x_3	0.92187500000000	−0.94531250000000	1.01171875000000
x_4	0.96679687500000	−0.98925781250000	1.01123046875000
x_5	0.98901367187500	−1.00012207031250	1.00555419921875
x_6	0.99728393554688	−1.00141906738281	1.00206756591797
x_7	0.99967575073242	−1.00087165832520	1.00059795379639
x_8	1.00013685226440	−1.00036740303040	1.00011527538300
x_9	1.00012606382370	−1.00012066960335	0.99999730288982

The second column in ε table is given by:

$\varepsilon_2^{(0)}$	0.75508945095620	−0.80690931523751	0.92412091301666
$\varepsilon_2^{(1)}$	0.98568372552246	−0.97021556689156	0.99226592068455
$\varepsilon_2^{(2)}$	1.00245975499866	−0.98906504956041	0.99330264728087
$\varepsilon_2^{(2)}$	1.00273344060741	−0.99818605148701	0.99772630543980
$\varepsilon_2^{(3)}$	1.00104013454301	−0.99992134445345	0.99944060495522
$\varepsilon_2^{(5)}$	1.00035441548961	−1.00009545867526	0.99987052159282
$\varepsilon_2^{(6)}$	1.00012178560030	−1.00007613375032	0.99997717407501
$\varepsilon_2^{(7)}$	1.00003710967910	−1.00004772657576	1.00000530844833

Finally, the fourth column in ε table is given by:

$\varepsilon_4^{(0)}$	1.00112801744667	−0.98874489258767	0.99310656004813
$\varepsilon_4^{(1)}$	1	−1	1
$\varepsilon_4^{(2)}$	1	−1	1
$\varepsilon_4^{(3)}$	1	−1	1
$\varepsilon_4^{(4)}$	1	−1	1
$\varepsilon_4^{(5)}$	1	−1	1

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