THE FIRST ORDER EVALUATIONS IN THE ASYMPTOTIC ANALYSIS OF SEQUENCES

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Abstract. We show how can be characterized the speed of convergence of the sequences by two sided estimations obtained staring from the first iterated limit.

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1. Introduction. A series of optimization problems conducts to some inequalities, asymptotic calculations, numerical calculations and approximations.

In the convergence problems it is important to know the speed of convergence (or "the order of convergence") which characterizes it.

So, we can consider several well-known examples.

(a) The convergence estimations of the Picard-Banach-Caccioppoli Theorem of fixed point of the contractions. Let (X,d) be a complete metric space and $f:X\to X$ be a contraction of constant $\alpha\in(0,1)$, i. e. such that

$$\mathrm{d}(f(x),f(y))\le\alpha\mathrm{d}(x,y)\qquad\text{for any}\quad x,y\in X.$$

Then, for the unique fixed point x^* of the contraction f we have the "à priori" error estimation

$$\mathrm{d}(x_n,x^*)\le\frac{\alpha^n}{1-\alpha}\cdot\mathrm{d}(x_0,x_1)\qquad n=0,1,2,\ldots$$

and the "à posteriori" error estimation

$$\mathrm{d}(x_n,x^*)\le\frac{\alpha}{1-\alpha}\cdot\mathrm{d}(x_{n-1},x_n).$$

(b) Let $f:[0,1]\to\mathbf{R}$ be a continuous function and B_nf the Bernstein polynomial of order n attached to f.

$$(B_nf)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right)\qquad x\in[0,1].$$

The Voronovskaya's theorems affirms that, if f is twice differentiable on $[0,1]$ then the following asymptotic result holds

$$\lim_{n\to\infty}n\left[(B_nf)(x)-f(x)\right]=\frac{x(1-x)}{2}f''(x).$$

(c) Many evaluations are given in the theory of approximation using the modulus of continuity, one of the most known result being an evaluation of Tiberiu Popoviciu

$$\left|(B_nf)(x)-f(x)\right|\le\frac{3}{2}\omega_f\left(\frac{1}{\sqrt{n}}\right).$$

Other evaluations are given in the theory of the approximative metods for solving equations, in the cuadrature formulas and many domains.

In the study of the functions of a discrete variable, i.e. of the sequences, these evaluations has a great importance.

So, the definition of the fundamental constant e, involving the formulas

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)=\mathrm{e}$$

is well-known, but the inequalities

$$\frac{\mathrm{e}}{2n+2}<\mathrm{e}-\left(1+\frac{1}{n}\right)^n<\frac{\mathrm{e}}{2n+1}$$

and

$$\frac{1}{n!(n+1)}<\mathrm{e}-\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)<\frac{1}{n!n}$$

show that the first convergence is slower than the second. As a consequence, the numerical computation of the number e uses the second one.

In the following we will analyze on example and after this we will present a selection of such results, especially obtained by us.

2. The analysis of an example. Consider the previously mentioned two sided estimation

$$\frac{\mathrm{e}}{2n+2}<\mathrm{e}-\left(1+\frac{1}{n}\right)^n<\frac{\mathrm{e}}{2n+1} \tag{2.1}$$

This inequality implies the relation

$$\lim_{n\to\infty} n\left(\mathrm{e}-\left(1+\frac{1}{n}\right)^n\right)=\frac{\mathrm{e}}{2}, \tag{2.2}$$

which is the first iterated limit of the sequence $(1+1/n)^n$ (respecting the sequence of functions atural variable $1/n, 1/n^2, 1/n^3, \ldots$). [But the equality (2.2) can be also obtained directly passing to the real variable and computing $\lim_{x\to\infty} x\left(\mathrm{e}-(1+1/x)^x\right)=\mathrm{e}/2$ (in certain cases it is necessary to apply firstly the lemma of Cesàro-Stolz see [5], [13] and after this to pass to the real variable)].

Conversely, the equality (2.2) may suggest as two sided estimation the following

$$\frac{\mathrm{e}}{2n+1}<\mathrm{e}-\left(1+\frac{1}{n}\right)^n<\frac{\mathrm{e}}{2n}.$$

But this is wrong! The correct inequality is (2.1).

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We will show here (following [16]) the technique to prove (2.1). The two sides of (2.1) must be proved separately; we must isolate the constant e and we obtain that the first part of (2.1) is equivalent to

$$u_n \stackrel{def}{=} \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n+1}\right)<\mathrm{e} \tag{2.3}$$

and the second part is equivalent to

$$\mathrm{e}<\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)\stackrel{def}{=} v_n. \tag{2.4}$$

Obviously

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \mathrm{e}\,,$$

then, to obtain (2.3) and (2.4) it is sufficient to prove that the sequence $(u_n)_n$ is strictly increasing and the sequence $(v_n)_n$ is strictly decreasing.

We pass to the positive real variable and we will prove that the functions that extend the previous sequences to the positive real variable, i. e.

$$u(x) \overset{def}{=} \left(1+\frac{1}{x}\right)^x \left(1+\frac{1}{2x+1}\right)$$

$$v(x) \overset{def}{=} \left(1+\frac{1}{x}\right)^x \left(1+\frac{1}{2x}\right)$$

has $u'(x) > 0$ and $v'(x) < 0$, for any $x > 0$. We obtain

$$u'(x) = \left(1+\frac{1}{x}\right)^x \left[\left(\ln(x+1) - \ln x - \frac{1}{x+1}\right)\frac{2x+2}{2x+1} - \frac{2}{(2x+1)^2}\right] \tag{2.5}$$

$$v'(x) = \left(1+\frac{1}{x}\right)^x \left[\left(\ln(x+1) - \ln x - \frac{1}{x+1}\right)\frac{2x+1}{2x} - \frac{1}{2x^2}\right] \tag{2.6}$$

We establish the sign of these derivatives using an inequality of J. Karamata

$$\frac{2}{2x+1} < \ln(x+1) - \ln x < \frac{1}{\sqrt{(x+1)}} \qquad (x>0) \tag{2.7}$$

(see [8], pag. 273). Applying the first part of (2.7) we obtain after a little calculation $u'(x) > 0$. To obtain $v'(x) < 0$ is sufficient to apply a relaxed form of the second part of (2.7) namely

$$\ln(x+1) - \ln x < \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x}\right)$$

and we obtain

$$v'(x) < -\frac{1}{4x^2(x+1)} \cdot \left(1+\frac{1}{x}\right)^x < 0\,.$$

These completes the proof.

3. A list of examples. The first iterated limit of many sequence suggest, each of them, a corresponding two sided estimation. We give some examples

- $\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1} - \mathrm{e}\right) = \frac{\mathrm{e}}{2}$ & $\frac{\mathrm{e}}{2n+1} < \left(1+\frac{1}{n}\right)^{n+1} - \mathrm{e} < \frac{\mathrm{e}}{2n}$ (see [14]) (3.1)
- $\lim_{n\to\infty} n\left(\frac{1}{\mathrm{e}} - \left(1-\frac{1}{n}\right)^n\right) = \frac{1}{2\mathrm{e}}$ & $\frac{1}{2n\mathrm{e}} < \frac{1}{\mathrm{e}} - \left(1-\frac{1}{n}\right)^n < \frac{1}{(2n-1)\mathrm{e}}$ (see [11]) (3.2)
- $\lim_{n\to\infty} n\left(\left(1-\frac{1}{n}\right)^{n-1} - \frac{1}{\mathrm{e}}\right) = \frac{1}{2\mathrm{e}}$ & $\frac{1}{(2n-1)\mathrm{e}} < \left(1-\frac{1}{n}\right)^{n-1} - \frac{1}{\mathrm{e}} < \frac{1}{(2n-2)\mathrm{e}}$ (see [11]) (3.3)
- $\lim_{n\to\infty} n(c_n - c) = \frac{1}{2}$ & $\frac{1}{2n+1} < c_n - c < \frac{1}{2n}$ (3.4)

where $c_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$ and $c = \gamma = \lim_{n\to\infty} c_n$ is the well-known constant of Euler (see[15]).

- $\lim_{n\to\infty} n^2(\gamma - R_n) = \frac{1}{2}$ & $\frac{1}{12(n+1)^2} < \gamma - R_n < \frac{1}{12n^2}$, (3.5)

where $R_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)$ (see[3]).

- $\lim_{n\to\infty} n^3(T_n - \gamma) = \frac{1}{24}$ & $\frac{1}{48(n+1)^3} < T_n\text{-}\gamma < \frac{1}{48n^3}$, (3.6)

where $T_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right)$ (see[10]).

- $\lim_{n\to\infty} n(\gamma - x_n) = \frac{1}{2}$ & $\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}$, (3.7)

where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n$ (see[21]).

The inequality (3.7) conducted C. Mortici and A. Vernescu to improve (3.4) as following

$$\frac{1}{2n} - \frac{1}{12n^2} < c_n - \gamma < \frac{1}{2n} - \frac{1}{12(n+1)^2} \tag{3.8}$$

(see [9]).

- $\lim_{n\to\infty} n^{s-1}(\zeta(s) - \zeta_n(s)) = \frac{1}{s-1}$ & $\frac{1}{(s-1)(n+1)^{s-1}} < \zeta(s) - \zeta_n(s) < \frac{1}{(s-1)n^{s-1}}$, (3.9)

where $s \in (1, \infty)$, $\zeta_n(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s}$ and $\zeta(s) = \lim_{n\to\infty} \zeta_n(s)$, is the famous function of Riemann (see [4]); a simple proof using only the sequences is given in [19].

- $\lim_{n\to\infty} n^{\alpha}(a_n(\alpha) - a(\alpha)) = \frac{1}{2}$ & $\frac{1}{\left(2 + \frac{1}{n}\right)n^{\alpha}} < a_n(\alpha) - a(\alpha) < \frac{1}{2n^{\alpha}}$, (3.10)

(see [1], [2]).

- $\lim_{n\to\infty} \Omega_n \sqrt{n} = \frac{1}{\sqrt{\pi}}$ & $\frac{1}{\sqrt{\pi\left(n + \frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi n}}$, (3.11)

where $\Omega_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 5 \cdot \ldots \cdot 2n}$ (see[8]).

Two stronger inequality are the following

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n + \frac{1}{4}\right)}} \quad \text{(see[8], [6])}$$

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n + \frac{1}{4}\right)}} \quad \text{(see[12])}$$

(see also [22] and [23] for other refinements and expansions).

- $\lim_{n\to\infty} n\left(\frac{\pi}{2} - W_n\right) = \frac{\pi}{8}$ & $\frac{\pi}{4(2n+1)}\left(1 - \frac{1}{8n}\right) < \frac{\pi}{2} - W_n < \frac{\pi}{4(2n+1)}$ (see [11]),(3.12)

where

$$W_n = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)}$$

is the sequence of the Wallis formula $\lim\limits_{n\to\infty} W_n = \frac{\pi}{2}$ (see [17]).

- $\lim\limits_{n\to\infty} n(\ln n - A) = \frac{1}{4}$ & $\frac{1}{4n+2} < \ln 2 - A_n < \frac{1}{4n+1}$, (3.13)

where $A_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \xrightarrow[(n\to\infty]{} \ln 2$ (see [18]).

- $\lim\limits_{n\to\infty} (-1)^n n(\ln 2 - S_n) = \frac{1}{2}$ & $\frac{1}{2n+2} < (-1)^n (\ln 2 - S_n) < \frac{1}{2n+1}$, (3.14)

where $S_n = 1 - \frac{1}{2} + \ldots (-1)^{n-1}\frac{1}{n}$ (see [20]).

- $\lim\limits_{n\to\infty} (-1)^n n\left(\frac{\pi}{4} - S_n\right) = \frac{1}{4}$ & $\frac{1}{4n+1} < (-1)^n \left(\frac{\pi}{4} - S_n\right) < \frac{1}{4}$, (3.15)

where $S_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots (-1)^{n-1}\frac{1}{2n-1}$ (see [20]).

- $\lim\limits_{n\to\infty} n\left(L_n - \frac{1}{\mathrm{e}}\right) = \frac{1}{2\mathrm{e}}$ & $\frac{1}{2\mathrm{e}n}\left(1 - \alpha\frac{\ln n}{n}\right) < L_n - \frac{1}{\mathrm{e}} < \frac{1}{2\mathrm{e}n}$, (3.16)

where $L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, $(n \geq 2)$, $\alpha > \frac{1}{4}$ (see [7]).

References

[1] D. Andrica, L. Tóth, *Ordinul de convergenţă al un şiruri de tipul* $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$, Astra Matematica, vol. I (1990), nr. 3 (iunie) 1990, 3-7.

[2] D. Andrica, V. Berinde, L. Tóth, A. Vernescu, *Ordinul de convergenţă al unor şiruri,* G. M. Vol. **103** (1998), nr. 7-8, 282-286.

[3] D. W. De Temple, *A quicker convergence to Euler' s constant,* A. M.M , vol. **100** (1993), 468-4570.

[4] G. M. Fihtenholt, *Curs de calcul diferenţial şi integral,* Ed. Tehnică, Bucureşti,

[5] H. G. Garnir, *Fonctions de variables réelles*, Tome I, Libraire Universitaire, Louvain & Gauthier-Villars, Paris, 1965.

[6] D. K. Kazarinoff, *On Wallis Formula*, Edinburgh, Math. Notes 40 (1956), 19-21.

[7] A. Lupaş, A. Vernescu, *Asupra unei conjecturi*, G. M. Seria A, vol. **19** (**98**) (2001), nr. 4, 212-217.

[8] D. S. Mitrinovć, P. M. Vasić, *Analytic Inequalities*, Springer Verlag, Berlin-New York-Heidelberg, 1970.

[9] C. Mortici, A. Vernescu, *An improvement of the convergence speed of the sequence* $(\gamma_n)_{n\geq 1}$ *converging to Euler's constant*, Analele St. ale Univ. "Ovidius" Constanta, Seria Mat. vol. **13** (2005), fasc. 1, 95-98.

[10] T. Negoi, *O convergenţă mai rapidă către constanta lui Euler,* G. M. Seria A, vol. **25** (**94**) (1997), nr. 2, 111-113.

[11] C. P. Niculescu, A. Vernescu, *Asupra ordinului de convergenţă al şirului* $\left(1 - \frac{1}{n}\right)^n$, G. M. vol. **109** (2004), nr. 4, 145-148.

[12] L. Panaitopol, *O rafinare a formulei lui Stirling*, G. M. vol. **90** (1985), nr. 9, 329-332.

[13] I. Rizzoli, *O teoremă Stolz-Cesàro*, G. M. vol. **95** (1990), nr. 10-11-12, 281-284.
[14] A. Vernescu, *O inegalitate privind numărul* e, G. M. vol. **87** (1982), nr. 2-3, 61-62.
[15] A. Vernescu, *Ordinul de convergenţă al şirului de definiţie al constantei lui Euler*, G. M. vol. **88** (1983), nr. 10-11, 380-381.
[16] A. Vernescu, *O demonstraţie simplă a unei inegalităţi relative la numărul* e, G. M. vol. **93** (1988), nr. 5-6, 206-207.
[17] A. Vernescu, *Ordinul de convergenţă al şirului din formula lui Wallis*, G. M. Seria A, vol. **12** (1991), nr. 1, 7-8.
[18] A. Vernescu, *Asupra convergenţei unui şir cu limita* ln 2, G. M. vol. **102** (1997), nr. 10-11, 370-374.
[19] A. Vernescu, *Asupra seriei armonice generalizate*, G. M. Seria A, vol. **15** (**104**) (1997), nr. 3, 186-190.
[20] *A. Vernescu, Rapiditatea de convergenţă a două serii celebre*, G. M. vol. **104** (1999), nr. 11, 421-426.
[21] A. Vernescu, *O nouă convergenţă accelerată către constanta lui Euler*, G. M. Seria A, vol. **17** (**96**) (1999), 273-278.
[22] A. Vernescu, *Sur l'approximation et le developpement asymptotique de la suite de terme général* $\frac{(2n-1)!!}{(2n)!!}$, Proc. of the Ann. Meet. of the Rom. Soc. of. Math. Sci. Bucharest 1997, may 29-june1, Tome 1, 205-213.
[23] A. Vernescu, *The natural proof of the inequalities of Wallis type*, Libertas Mathematica, vol. **24** (2004), 183-190.