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# UNIQUE FIXED POINT FOR A SEMILINEAR OPERATOR

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***Abstract:*** *Let $H$ be a real Hilbert space. We consider the semilinear operator $S = T - A$, where $T : H \to H$ is a strict contraction and $A : H \to H$ is a linear, continuous and positive operator. It is proved that the operator $S$ has an unique fixed point.*

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## 1 Introduction

Let $H$ be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the norm $||\cdot||$. Let $T : H \to H$ be a strict contraction, i. e.

$$||T(x) - T(y)|| \leq \alpha||x - y|| \text{ for all } x, y \in H \text{ with } \alpha \in (0, 1).$$

It is a very well known fact that $T$ has an unique fixed point( due to the famous Banach Fixed Point Theorem). We are interested in finding operators $V : H \to H$, so that the operators $T - V$ or $T + V$ could also have "the property of the unique fixed point". In this paper we will prove that the previous problem is a well posed problem.

Using the Cauchy-Schwarz inequality, we have

$$< (I - T)x - (I - T)y, x - y >= ||x - y||^2 - < T(x) - T(y), x - y > \geq$$

$$\geq ||x - y||^2 - ||T(x) - T(y)|| \cdot ||x - y||$$

for all $x, y \in H$ ( $I$ is the identity of $H$).

Consequently $I - T$ is strongly monotone, i. e.

$$< (I - T)x - (I - T)y, x - y >\geq (1 - \alpha)||x - y||^2 \text{ for all } x, y \in H, \tag{1}$$

with $1 - \alpha > 0$.

## 2 The result

**Theorem** *Let $H$ be a real Hilbert space and $T : H \to H$ be a strict contraction. If $A : H \to H$ is linear, continuous and positive, then the operator $S = T - A$ has an unique fixed point.*

**Proof:** Let $B : H \to H$ defined by $Bu = (I - T + A)u$. Using (1) and the positivity of $A$ we obtain that

$$< B(x) - B(y), x - y >=< (I - T)x - (I - T)y, x - y > + < A(x - y), x - y >$$

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$$\geq (1-\alpha)||x-y||^2 \text{ for all } x, y \in H.$$

Also, for every $x$, $y \in H$, we have

$$||B(x)-B(y)|| \leq ||x-y|| + ||T(x)-T(y)|| + ||Ax-Ay|| \leq$$

$$\leq (1+\alpha+||A||)||x-y||.$$

Hence $B$ is a strongly monotone Lipschitz operator.

Let us consider now, for $\gamma > 0$, the operator $N_\gamma : H \to H$, defined by

$$N_\gamma u = (I-\gamma B)u.$$

We have

$$||N_\gamma x - N_\gamma y||^2 =< x-\gamma Bx-(y-\gamma By), x-\gamma Bx-(y-\gamma By) >=$$

$$= ||x-y||^2 - 2\gamma < Bx-By, x-y > +\gamma^2||Bx-By||^2 \leq$$
$$\leq [1-2\gamma(1-\alpha)+\gamma^2(1+\alpha+||A||)^2] \cdot ||x-y||^2,$$

so

$$||N_\gamma x - N_\gamma y|| \leq \sqrt{1-2\gamma(1-\alpha)+\gamma^2(1+\alpha+||A||)^2} \cdot ||x-y||,$$

for all $x,\ y \in H$. Further, remark that if

$$\gamma \in \left(0, \frac{2(1-\alpha)}{(1+\alpha+||A||)^2}\right),$$

then $N_\gamma$ is a strict contraction, because

$$\sqrt{1-2\gamma(1-\alpha)+\gamma^2(1+\alpha+||A||)^2} < 1$$

and consequently, $N_\gamma$ has an unique fixed point in $H$. In other words, it exists an unique element $u^* \in H$ such that

$$u^* = N_\gamma u^*,$$

which is succesively equivalent with

$$u^* = (I-\gamma B)u^* \Leftrightarrow u^* = u^* - \gamma Bu^* \Leftrightarrow Bu^* = 0.$$

Further,

$$Bu^* = 0 \Leftrightarrow (I-T+A)u^* = 0 \Leftrightarrow u^* = (T-A)u^*,$$

thus $u^*$ is the fixed point of $T-A$ and the proof of the theorem is complete.□

## References


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