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A CONSTRUCTION OF AN ORTHOGONAL BASIS IN SOME SOBOLEV SPACES

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Abstract: *It is used the method of orthogonal sequences of Bergmann [2], [4] and Vekua [5], to find an orthogonal basis in the Sobolev space $H_0^1(D)$, where D is a quarter of circle. The elements of the basis are the solutions of some eigenvalue boundary problem.*

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2000 Mathematics subject classification: *35C10, 65M06*

1 Introduction

In practise arise real difficulties in the problem of finding a base in Hilbert spaces. We give here a method of elimination of these difficulties using Bergmann's method of double orthogonal sequences [2], [3], [4].

Let $(H, (\cdot,\cdot))$, $(V, <\cdot,\cdot>)$ be real, separable Hilbert spaces and denote by $\|\cdot\|$, $|\cdot|$ the corresponding norms, respectively. In what follows, we use the next result due to Bergmann [2]:

Theorem 1. *Assume that $H \subset V$ and the imbedding $H \hookrightarrow V$ is compact,*

$$|x| \leq c\,\|x\|$$

for every $x \in H$ and for some positive constant c. Then there exist an increasing, unbounded sequence $(\lambda_n)_{n\geq 1}$ of positive reals and a sequence $(e_n)_{n\geq 1} \subset H$ which is orthogonal with respect to both inner products, i.e.

$$(e_m, e_n) = \lambda_n \delta_{mn} \quad , \quad < e_m, e_n >= \delta_{mn}\,, \tag{1.1}$$

for all positive integers m, n. Moreover, $(e_n)_{n\geq 1}$ is complete in H.

Theorem 1 is in fact a method to find an orthogonal basis in H where the elements of the basis are the solutions of some optimization problems. Remark that from (1.1), we can derive the equalities

$$(e_m, e_n) = \lambda_n < e_m, e_n >,$$

for all $m, n \geq 1$. Because of completness of the system $(e_n)_{n\geq 1}$, it follows that

$$(e_n, v) = \lambda_n < e_n, v >, \tag{1.2}$$

for every $n \geq 1$, $v \in H$. In consequence, the elements of the orthogonal basis $(e_n)_{n\geq 1}$ can be considered as the solutions of the eigenvalue problem (1.2). In fact, this is an useful method to find a basis in a real separable Hilbert space, as we can see below.

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The set of all functions $u \in L^2(D)$ with $u = 0$, on ∂D, having generalized derivative is denoted by $H_0^1(D)$. The space $H_0^1(D)$ also called Sobolev space is a Hilbert space relative to the scalar product

$$(u,v) = \int_D uv + \int_D \nabla u \nabla v \ , \quad u,v \in H_0^1(D)$$

with the corresponding norm

$$\|u\| = \left(\int_D u^2 + \int_D |\nabla u|^2 \right)^{1/2} \ , \quad u \in H_0^1(D).$$

Consider also the Hilbert space $L^2(D)$ endowed with the usual scalar product

$$< u,v >= \int_D uv \ , \quad u,v \in L^2(D)$$

and the usual norm

$$|u| = \left(\int_D u^2 \right)^{1/2} \ , \quad u \in L^2(D).$$

The imbedding

$$H_0^1(D) \hookrightarrow L^2(D)$$

is compact because

$$|u| \leq \|u\| \ , \ for\ every \ \ u \in H_0^1(D).$$

In order to give a method to find an orthogonal basis in $H_0^1(D)$, we will use Theorem 1. The eigenvalue problem (1.2) can be written as

$$\int_D e_n v + \int_D \nabla e_n \nabla v = \lambda_n \int_D e_n v \ , \ for\ every \ \ v \in H_0^1(D), \ n \geq 1. \tag{1.3}$$

But $v = 0$, on ∂D, so

$$\int_D \nabla e_n \nabla v = - \int_D v \Delta e_n,$$

if e_n is twice derivable. Hence (1.3) is equivalent with

$$\int_D e_n v - \int_D v \Delta e_n = \lambda_n \int_D e_n v,$$

or

$$\int_D (\Delta e_n + (\lambda_n - 1) e_n) v = 0 \ , \ for\ every \ \ v \in H_0^1(D).$$

We deduce that $(e_n)_{n \geq 1}$ are the eigenfunctions of the next boundary problem

$$\begin{cases} -\Delta u(x) = (\lambda - 1) u(x) & in \ \ D \\ u(x) = 0 & on \ \ \partial D \end{cases} \tag{1.4}$$

2 The Result

Let $D \subset \mathbb{R}^2$ be a domain having the boundary with a corner of angle π/k, where $k \geq 1/2$. A method for finding the eigenfunctions for the laplacian is Bergman-Vekua method (e.g. [2]) which gives

$$u(r,t) = \sum_{j=1}^{N} c_j J_{jk}(\sqrt{\lambda} r) \sin jkt, \tag{2.1}$$

where (r,t) are polar coordinates, J_β is Bessel function of order β and c_j and λ will be determined.

We will consider the domain D as the quarter of the circle $(k=2)$ with radius $r=1$. For sake of simplicity we will impose the boundary conditions only in two distinct points $P_i(1,t_i)$ with $i=\overline{1,2}$(in polar coordinates).

In this case, $N=2$ and the solution is given by the formula

$$u(r,t)=\sum_{j=1}^{2} c_j J_{2j}(\sqrt{\lambda}r)\sin 2jt. \tag{2.2}$$

The condition (2.2) with $r=1$, and $t_i \in (0,\pi/2)$ can be equivalently written as

$$\begin{cases} c_1J_2(\sqrt{\lambda})sin(2t_1)+c_2J_4(\sqrt{\lambda})sin(4t_1)=0 \\ c_1J_2(\sqrt{\lambda})sin(2t_2)+c_2J_4(\sqrt{\lambda})sin(4t_2)=0 \end{cases}.$$

Now we are interested in finding nontrivial solutions (c_1,c_2), so the attached determinant must vanishes,

$$\begin{vmatrix} J_2(\sqrt{\lambda})sin(4t_1) & J_4(\sqrt{\lambda})sin(4t_1) \\ J_2(\sqrt{\lambda})sin(4t_2) & J_4(\sqrt{\lambda})sin(4t_2) \end{vmatrix}=0$$

Hence we must have

$$J_2(\sqrt{\lambda})J_4(\sqrt{\lambda})\begin{vmatrix} J_2(\sqrt{\lambda})sin(4t_1) & J_4(\sqrt{\lambda})sin(4t_1) \\ J_2(\sqrt{\lambda})sin(4t_2) & J_4(\sqrt{\lambda})sin(4t_2) \end{vmatrix}=0$$

therefore:

$$J_4(\sqrt{\lambda})J_8(\sqrt{\lambda})sin(4t_1)sin(4t_2)sin(2(t_1-t_2))sin(2(t_1+t_2))=0$$

Because $t_1\neq t_2$ and t_1, $t_2\in(0,\pi/4)$ all the above sinuses are non-zero, therefore we have:

$$J_4(\sqrt{\lambda})J_8(\sqrt{\lambda})=0. \tag{2.6}$$

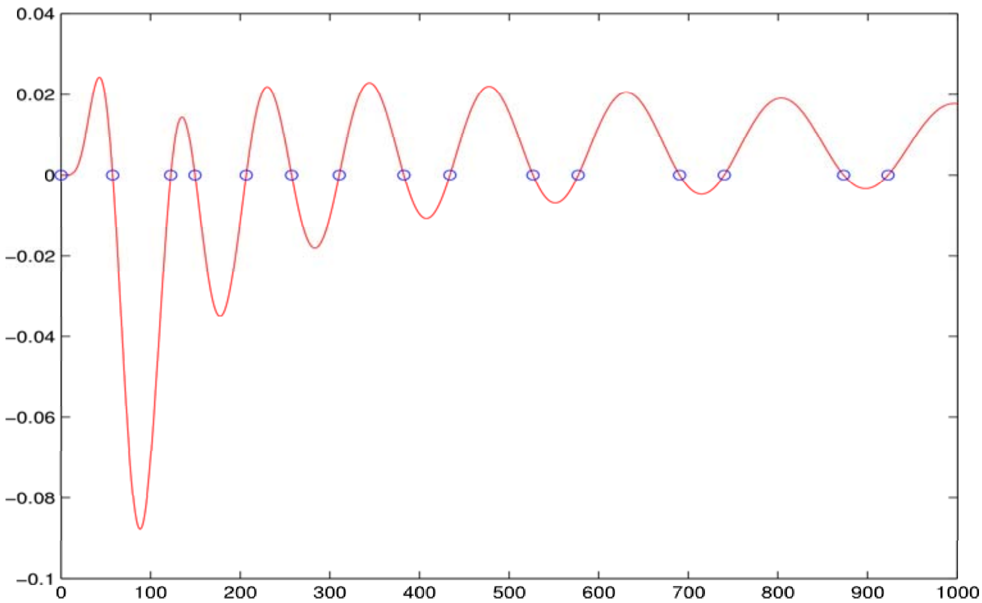

Figure 1: Graphic of $J_4(\sqrt{\lambda})J_8(\sqrt{\lambda})$

For the estimation of the approximative solutions we use the assymptotic expansion of the Bessel functions as in Ikonomou,Köhler and Jacob. Let $\nu \in \mathbf{R}$ and let $p \in \mathbf{N}$ such that $\nu - p \leq \frac{1}{2}$. We approximate:

$$J_\nu(x) \simeq J_{\nu,p}(x)$$

where

$$J_{\nu,p}(x) = \frac{1}{\sqrt{2\pi x}} \sum_{i=1}^{2} e^{\sigma_k i \varphi_\nu(x)} P_k(x,\nu,p) \text{ with } \varphi_\nu(x) = x - \frac{\nu\pi}{2} - \frac{\pi}{4}$$

$$P_k(x,\nu,p) = \sum_{m=0}^{m-1} \frac{a_{\nu,m}}{(2\sigma_k i x)^m} \text{ with } \sigma_1 = 1, \qquad \sigma_2 = -1$$

and

$$a_{\nu,m} = \frac{(1/2-\nu)_m (1/2+\nu)_m}{m!} \text{ with } (x)_m = x(x+1)(x+2)\dots(x+m+1)$$

According to Ikonomou,Köhler and Jacob by denoting

$$\varepsilon(x) = J_\mu(x)J_\nu(x) - J_{\mu,q}(x)J_{\nu,p}(x) \text{ where } \nu - p \leq \frac{1}{2} \text{ and } \mu - q \leq \frac{1}{2}$$

we have the following estimation:

$$\mid \varepsilon(x) \mid = \mathbf{O}(x^{-1-min(p,q)}) \text{ for } x \longrightarrow \infty$$

Also is demonstrated that for $p \geq 6$ and $x \geq 5\pi$ we have $\mid \varepsilon(x) \mid \leq 10^{-5}$.

Here are the roots between 0 and 100 calculated with a precision of 10^{-4}

0 57.583 122.43 149.45 206.57 257.21 310.32 382.38 433.76 526.48 576.91 689.95 739.79 872.95 922.4

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