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PARTIALLY ORDERED SETS AND APPLICATIONS TO THE RECURSIVE AND THE INVERSION FORMULAS OF MÖBIUS

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Abstract: *In this paper we present some examples of partially ordered sets and also some applications of the inversion formula of Möbius and recursive formula of Möbius.*

1 Preliminary notions about partially ordered sets

Definition 1. A partially ordered set (or *poset* for short) is an ordered pair $(P, \leq)$, consisting of a set P and a relation $\leq$ on P satisfying the following properties:

1. for all $x \in P$, $x \leq x$ (reflexivity);
2. for all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$(anti-symmetry);
3. for all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Remark 1. Obviously the notation $x < y$ is used when both $x \leq y$ and $x \neq y$ holds, and when it is ambiguous to which poset the relation belongs, we will write $\leq_P$ instead of $\leq$.

Example 1. The following are standard examples of posets. Given $m, n \in \mathbf{N}$ we define $[m, n] := \{m, m+1, m+2, ..., n\}$, if $m = 1$, let $[n] = [1, n] = \{1, 2, 3, .., n\}$.

1. The sets **N**, **Z**, **Q** and **R** are all posets, together with their linear orderings.
2. Given $n \in \mathbf{N}$, the poset $\mathbf{n}$ is the set $[n]$ ordered by linear relation $1 < 2 < ... < n$.
3. Given $n \in \mathbf{N}$, the poset D_n is the set of all positive divisors of n ordered by divisibility, i.e. $x \leq_{D_n} y$ if and only if x divides y.
4. Given $n \in \mathbf{N}$, the poset B_n is the power set of $[n]$ ordered by inclusion, i.e. $X \leq_{B_n} Y$ if and only if $X \subseteq Y$.

Definition 2. If $x \leq y$in the poset P, then the *closed interval* (or *interval* for short) from x to y , denoted by $[x, y]$ is the set $[x, y] := \{z \in P / x \leq z \leq y\}$. The open interval from x to y is the set $(x, y) := \{z \in P / x < z < y\}$. The collection of all intervals of P is denoted $Int(P)$.

Remark 2. Notice that $[x, x] = \{x\}$ and $(x, x) = \phi$.

Definition 3. A poset P is *locally finite* if every interval of P is finite.

Definition 4. If $x < y$ and $(x, y) = \phi$ in the poset P, then we say that y *covers* x (i.e. between x and y there is no other element from P).

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Definition 5. Two elements x and y in the poset P are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable.*

Definition 6. A poset P is a *chain* (or *totally ordered set* or *linearly ordered set*) if every pair of elements is comparable and the *length* of a chain is equal with the number of elements composing the chain minus 1.

Definition 7. The *Hasse Diagram* of a finite poset P is the graph whose vertex set is P and whose edge set is the covering relations in P. If x covers y in P , then x is drown with a higer horizontal coordinate than y.

2 The incidence algebra of a locally finite poset

In this section we introduce an algebraic structure for locally finite posets, as in [2] and [5], that is useful in answering many of the combinatorial questions associated with such poset.

Let $(P, \leq)$ be a poset and k a field of characteristic zero. Let

$$A_k(P) = \{f : P \times P \to k \,/\, if\, not\, x \leq y\, then\, f(x,y) = 0\}.$$

When is not ambiguous we can denote with $A(P)$ instead $A_k(P)$.

The set $A_k(P)$becomes a k – vector space, where for $f, g \in A_k(P)$ and $\alpha \in k$ we have:

$$(f+g)(x,y) = f(x,y) + g(x,y)$$

$$(\alpha \cdot f)(x,y) = \alpha f(x).$$

We define the *convolution* $f * g$ for all $f, g \in A_k(P)$ such that
$(f*g)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$, if $x \leq y$ and
$(f*g)(x,y) = 0$ else.

Remark 3. Notice that since P is locally finite, the number of summands in the above sum is finite. Therefore, convolution is well-defined. For all $f \in A_k(P)$ and $n \in \mathbf{N}$, denote the convolution of f with itself n times by f^n.

Theorem 1. The set $A_k(P)$ with the operations: addition, multiplication with scalars and the convolution is an algebra called *incidence algebra* of the poset $(P, \leq)$. The elements of $A_k(P)$ we call *incidence functions.*

Proof. (1) (Convolution is associative) Since the number of summands is finite, we have:

$$[(f*g)*h](x,y) = \sum_{x \leq z \leq y} (f*g)(x,z) \cdot h(z,y) =$$

$$= \sum_{x \leq z \leq y} [\sum_{x \leq w \leq z} f(x,w) \cdot g(w,z)] \cdot h(z,y) = \sum_{x \leq w \leq y} f(x,w) \cdot [\sum_{w \leq z \leq y} g(w,z) \cdot h(z,y)] =$$

$$= \sum_{x \leq w \leq y} f(x,w) \cdot (g*h)(w,y) = [f*(g*h)](x,y).$$

(2) (Convolution is left distributive) Since multiplication in k is left distributive, we have:

$$[f*(g+h)](x,y) = \sum_{x \leq z \leq y} f(x,z) \cdot (g+h)(z,y) = \sum_{x \leq z \leq y} f(x,z) \cdot [g(z,y) + h(z,y)] =$$

$$= \sum_{x \leq z \leq y} [f(x,z) \cdot g(z,y) + f(x,z) \cdot h(z,y)] = \sum_{x \leq z \leq y} f(x,z) \cdot g(z,y) + \sum_{x \leq z \leq y} f(x,z) \cdot h(z,y) =$$

$$= (f * g)(x, y) + (f * h)(x, y) = [(f * g) + (f * h)] (x, y).$$

(3) (Convolution is right distributive) Since multiplication in k is right distributive, we have:

$$[(f + g) * h](x, y) = \sum_{x \leq z \leq y} (f + g)(x, z) \cdot h(z, y) = \sum_{x \leq z \leq y} [f(x, z) + g(x, z)] \cdot h(z, y) =$$

$$= \sum_{x \leq z \leq y} [f(x, z) \cdot h(z, y) + g(x, z) \cdot h(z, y)] = \sum_{x \leq z \leq y} f(x, z) \cdot h(z, y) + \sum_{x \leq z \leq y} g(x, z) \cdot h(z, y) =$$

$$= (f * h)(x, y) + (g * h)(x, y) = [(f * h) + (g * h)] (x, y).$$

(4) (Convolution identity) Let $\delta \in A_k(P)$ be defined for all $x, y \in P$ such that

$$\delta(x, y) = \begin{cases} 1 , & if\ x = y \\ 0 , & if\ x \neq y \end{cases} .$$

Then $(f * \delta)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot \delta(z, y) = f(x, y)$.

Also, $(\delta * f)(x, y) = \sum_{x \leq z \leq y} \delta(x, z) \cdot f(z, y) = f(x, y)$.

Therefore, δ is an identity for $A_k(P)$ under convolution.

Finally, we have that $A_k(P)$ is a ring with identity.

Let $\varphi : k \to A_k(P)$ be defined by $\varphi(a) := \delta_a := a \cdot \delta$, $\forall a \in k$.

For $\forall a, b \in k$ we have $\varphi(a+b) = \delta_{a+b} = (a+b) \cdot \delta = (a \cdot \delta) + (b \cdot \delta) = = \delta_a + \delta_b = \varphi(a) + \varphi(b)$. Also, $\varphi(ab) = \delta_{ab} = (ab) \cdot \delta = (a \cdot \delta) * (b \cdot \delta) = \delta_a * \delta_b = = \varphi(a) * \varphi(b)$. Thus φ is a ring homomorphism.

Since multiplication in k is commutative, it follows from the above that $\delta_a * \delta_b = (ab) \cdot \delta = (ba) \cdot \delta = \delta_b * \delta_a$ so that $\varphi(k) \subseteq Z(A_k(P))$. Because $\delta_{1_k} = 1_k \cdot \delta = \delta$, we have $\varphi(1_k) = 1_{V(Int(P),k)}$. Finally we obtain that $(A_k(P), \varphi)$ is an algebra over k.

Theorem 2. Let $f \in A(P)$. The following conditions are equivalent:

1. f has a left inverse;
2. f has a right inverse;
3. f has a two-sided inverse;
4. for all $x \in P$, $f(x, x) \neq 0$.

If f has any inverse, then it is the unique two-sided inverse of f.

Proof. Assume $f \in I(P)$.

(1⇔4) Suppose $g \in I(P)$ is a left inverse of f, i.e. $g * f = \delta$. This is true if and only if:

$$g(x, y) = \begin{cases} f(x, x)^{-1} & if\ \ x = y \\ -f(y, y)^{-1} [\sum_{x \leq z \leq y} g(x, z) f(z, y)] & if\ \ x < y \end{cases} ,$$

the second case is due to the fact that $(g * f)(x, y) = \sum_{x \leq z < y} g(x, z) f(z, y) + g(x, y) f(y, y)$. Thus g exists if and only if for all $x \in P$, we have $f(x, x) \neq 0$.

(2⇔4) Suppose $h \in I(P)$ is a right inverse of f . Similar to the above argument:

$$h(x, y) = \begin{cases} f(x, x)^{-1} & if\ \ x = y \\ -f(x, x)^{-1} \left[\sum_{x < z \leq y} f(x, z) h(z, y)\right] & if\ \ x < y \end{cases} ,$$

so that h exists if and only if for all $x \in P$, we have $f(x,x) \neq 0$.

(3⇔4) A two-sided inverse is necessarily a left and right inverse. Thus this result follows from the previous arguments.

Let's prove the unique of this inverse.

Suppose that g is a left inverse of f . Then the theorem implies the existence of a right inverse of f , let it be h, so $g * f = \delta = f * h$. The theorem also provides a two-sided inversef' of f. Then $g = g * \delta = = g * (f * f') = (g * f) * f' = \delta * f' = f' = f' * \delta =$

$= f' * (f * h) = (f' * f) * h = \delta * h = h$. Therefore, $g = f' = h$, proving that any inverse is two-sided and unique.

3 Some incidence functions

Through this section we will assume that P is a locally finite poset. Of all the functions from $A_k(P)$, there are a few of particular interest.

Delta function

We already saw that *delta function* δ is the identity of $A_k(P)$ and is defined for all $x, y \in P$ such that

$$\delta(x,y) = \begin{cases} 1\ , & if\ x = y \\ 0\ , & if\ x \neq y \end{cases} .$$

Also we have that $\delta_a = a \cdot \delta\ ,\ \forall a \in k$.

Zeta and Chain functions

Another important function from the incidence algebra $A_k(P)$ is *zeta function* ζ, defined such that: $\zeta(x,y) := 1$, $\forall x \leq y$ in P.

Then for all $x \leq y$ from P we have $\zeta^2(x,y) = \sum_{x \leq z \leq y} \zeta(x,z) \cdot \zeta(z,y) = = \sum_{x \leq z \leq y} 1 \cdot 1 = \sum_{x \leq z \leq y} 1 = |[x,y]|$. Therefore, $\zeta^2(x,y)$ counts the numbers of elements in $[x,y]$.

Notice that:

$$(\zeta - \delta)(x,y) := \begin{cases} 1 - 1 = 0 & if\ \ x = y \\ 1 - 0 = 1 & if\ \ x < y \end{cases}$$

and

$$(\zeta - \delta)^2(x,y) = \sum_{x \leq z \leq y} (\zeta - \delta)(x,z) \cdot (\zeta - \delta)(z,y) = \sum_{x < z < y} (\zeta - \delta)(x,z) \cdot (\zeta - \delta)(z,y) = \sum_{x < z < y} 1$$

which is equal to the number of chains of $[x,y]$ of length 2.

By induction it follows that $(\zeta - \delta)^k(x,y)$ counts the number of chains of $[x,y]$ of length k. Thus $\eta := \zeta - \delta$ is the *chain function.*

A sequence of functions $f_1, f_2, f_3, ...$ of $A_k(P)$ *converges* to a function $f \in A_k(P)$ if for all intervals $[x,y]$ from P, does exist an $N \in \mathbf{N}$ such that for every $n \geq N$, $f_n(x,y) = f(x,y)$. This defines a topology on $A_k(P)$.

Now consider:

$$(\delta_2 - \zeta)(x,y) := \begin{cases} 2 - 1 = 1 & if\ x = y \\ 0 - 1 = -1 & if\ x < y \end{cases} .$$

From the theorem above result that $\delta_2 - \zeta$ have an inverse:

$$(\delta_2 - \zeta)^{-1} = (\delta - (\zeta - \delta))^{-1} = \sum_{k=0}^{\infty} \eta^k,$$

because $\sum_{k=0}^{\infty}\eta^k$ converges in $A_k(P)$. Therefore, because of the interpretation of η we obtain that $(\delta_2-\zeta)^{-1}(x,y)$ counts the number of all chains from the interval $[x,y]$.

Lambda and Cover functions

Another function from the incidence algebra $A_k(P)$ is *lambda function* λ defined for every interval $[x,y]$ such that:

$$\lambda(x,y):=\begin{cases}1 \ \ if \ \ x=y \ or \ y \ covers \ x\\ 0 \ \ else\end{cases}.$$

We observe that:

$$(\lambda-\delta)(x,y):=\begin{cases}1 \ \ if \, y \ covers \ x\\ 0 \ \ else\end{cases}.$$

Let's denote$\kappa:=\lambda-\delta$ and we call it *cover function.*

We have $\kappa^2(x,y)=\sum_{x\le z\le y}\kappa(x,z)\cdot\kappa(z,y)$ and because $\kappa(x,z)\cdot\kappa(z,y)\neq 0$ only if y covers z, means that the chain $x<z<y$ is saturated. Because the interval $[x,y]$ is a finite one the condition that the chain $x<z<y$ to be saturated is equivalent with that to be a maximal chain. Finally, $\kappa^2(x,y)$ counts the number of maximal chains of length 2 from the interval $[x,y]$. In conclusion we have that $\kappa^k(x,y)$ give us the number of maximal chains of length k in the interval $[x,y]$.

Now consider

$$(\delta-\kappa)(x,y):=\begin{cases}1-0=1 \ \ \ if \ \ x=y\\ 0-1=-1 \ \ if \ \ y \ covers \ x\\ 0-0=0 \ \ \ else\end{cases}.$$

From the theorem above it results that $\delta-\kappa$ have an inverse: $(\delta-\kappa)^{-1}=\sum_{k=0}^{\infty}\kappa^k$, the sum is valid because $\sum_{k=0}^{\infty}\kappa^k$ converges in $A_k(P)$ for every interval $[x,y]$ in P. We obtain that $(\delta-\kappa)(x,y)$ gives us the number of all maximal chains in $[x,y]$.

Möbius function.

From the theorem we obtain that zeta function ζ possesses an inverse in $A_k(P)$.

The Möbius function is $\mu:=\zeta^{-1}$.

The relation $\mu*\zeta=\delta$ is equivalent to the following *recursive formula* of μ:

$$\mu(x,y):=\begin{cases}1 & if \ \ x=y\\ -\sum_{x\le z<y}\mu(x,z) & if \ \ x<y\end{cases},$$

since $(\mu*\zeta)(x,y)=\sum_{x\le z\le y}\mu(x,z)\cdot\zeta(z,y)=\sum_{x\le z\le y}\mu(x,z)\cdot 1=\sum_{x\le z<y}\mu(x,z)+\mu(x,y)=\delta(x,y)$

Example 2. 1. In case of the local finite poset of the positive integers ordered by divisibility, we have $\mu(a,b)=\mu\left(\frac{b}{a}\right)$, where the second μ is the classical Möbius function, introduced in number theory in the 19-th century.

2. The set of all subsets of E ordered by inclusion is a poset locally finite. In this case the Möbius function is $\mu(S,T)=(-1)^{|T\setminus S|}$, where S and T are finite subsets of E such that $S\subseteq T$.

3. The Möbius function for the set of positive integers with the usual order is $\mu(x,y)=\begin{cases}1 \ \ if \ \ y-x=0\\ -1 \ \ if \ \ y-x=1\\ 0 \ \ if \ \ y-x>1\end{cases}$. This corresponds to the sequence $(1,-1,0,0,0,...)$ of coefficients from the formal series $1-z$, and zeta function ζ corresponds to the sequence $(1,1,1,...)$ of coefficients from $(1-z)^{-1}=1+z+z^2+z^3+....$ Delta δ function in this incidence algebra corresponds to the formal series 1.

4. Now, let consider the poset of all partition of a finite set and denote $\sigma\le\tau$ if σ is a finer partition than τ. Then, the Möbius function is: $\mu(\sigma,\tau)=(-1)^{n-r}(2!)^{r_3}(3!)^{r_4}...((n-1)!)^{r_n}$,

where n is the number of elements of the finer partition σ, r is the number of elements of $\tau$and r_i is the number of the sets from τ which contain exactly i elements from σ.

4 The inversion formula of Möbius

Theorem 3. Let $(P, \leq)$ be a poset and G an commutative group. Also, let $f, g, h : P \to G$. Then:

1. $g(x) = \sum_{x \leq y} f(y) \Leftrightarrow f(x) = \sum_{x \leq y} \mu(x, y) g(y)$;
2. $h(x) = \sum_{y \leq x} f(y) \Leftrightarrow f(x) = \sum_{y \leq x} \mu(x, y) h(y)$.

The proof is in [5].

5 Applications

In this section we present some useful applications to the recursive formula and also the inversion formula Möbius.

5.1. *Remark 4.* For any finite poset P does exist a counting $P = \{x_1, x_2, ..., x_n\}$ such that if $x_i < x_j$ then $i < j$.

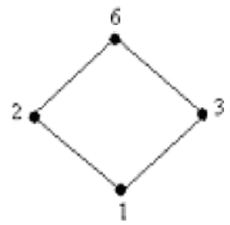
Suppose that P is $P = \{x_1, x_2, ..., x_n\}$. Then we can associate to zeta function $\zeta : P \times P \to \mathbf{Z}$, $\zeta(x, y) := 1$, $\forall x \leq y$ in P, a matrix $A \in M_n(\mathbf{Z})$ like:

$A = \begin{pmatrix} 1 & \zeta_{12} & \zeta_{13} & ... & \zeta_{12} \\ 0 & 1 & \zeta_{23} & ... & \zeta_{2n} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ where $\zeta_{ii} = 1, \forall i = \overline{1, n}$.

Because det $A = 1 \neq 0$, then A is invertible in $M_n(\mathbf{Z})$.

We know that the Möbius function $\mu := \zeta^{-1}$ is the inverse of zeta function, then the matrix $B = (\mu_{ij})_{i,j=\overline{1,n}}$ is the inverse of A.

Example 3. The poset $D_6 = \{1, 2, 3, 6\}$ with the partial order given by divisibility have the Hasse diagram:



We have the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for the zeta function of D_6, and $B = A^{-1} =$

$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the matrix associated to the Möbius function of D_6.

This is due to the recursive formula of μ:

$$\mu(1, 1) = 1; \mu(2, 2) = 1; \mu(3, 3) = 1; \mu(4, 4) = 1$$

$$\mu(1, 2) = -\mu(1, 1) = -1$$

$$\mu(1,3) = -\mu(1,1) = -1$$

$$\mu(1,6) = -\mu(1,1) - \mu(1,2) - \mu(1,3) = -1 + 1 + 1 = 1$$

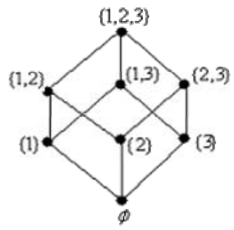
$$\mu(2,1) = 0, \mu(3,1) = 0; \mu(3,2) = 0; \mu(4,1) = 0; \mu(4,2) = 0; \mu(4,3) = 0;$$

$\mu(2,3) = 0$because in D_6;

$$\mu(2,6) = -\mu(2,2) = -1;$$

$$\mu(3,6) = -\mu(3,3) = -1.$$

Example 4. The poset $B_3 = \{\phi; \{1\}; \{2\}; \{3\}; \{1,2\}; \{1,3\}; \{2,3\}; \{1,2,3\}\}$ with the partial order given by inclusion have the Hasse diagram:



We have the matrix $A = \begin{pmatrix} 1&1&1&1&1&1&1&1\\ 0&1&0&0&1&1&0&1\\ 0&0&1&0&1&0&1&1\\ 0&0&0&1&0&1&1&1\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$ for the zeta function of B_3,

and $B = A^{-1} = \begin{pmatrix} 1&-1&-1&-1&1&1&1&-1\\ 0&1&0&0&-1&-1&0&1\\ 0&0&1&0&-1&0&-1&1\\ 0&0&0&1&0&-1&-1&1\\ 0&0&0&0&1&0&0&-1\\ 0&0&0&0&0&1&0&-1\\ 0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$ is the matrix associated to the Möbius function of B_3. This is due to the recursive formula of μ:

$$\mu(\phi,\phi) = \mu(\{1\},\{1\}) = \mu(\{2\},\{2\}) = \mu(\{3\},\{3\}) = \mu(\{1,2\},\{1,2\}) = \mu(\{1,2\},\{1,2\}) =$$

$$= \mu(\{1,3\},\{1,3\}) = \mu(\{2,3\},\{2,3\}) = \mu(\{1,2,3\},\{1,2,3\}) = 1;$$

$$\mu(\{1\},\phi) = \mu(\{2\},\phi) = \mu(\{2\},\{1\}) = \mu(\{3\},\phi) = \mu(\{3\},\{1\}) = \mu(\{3\},\{2\}) =$$

$$= \mu(\{1,2\},\phi) = \mu(\{1,2\},\{1\}) = \mu(\{1,2\},\{2\}) = \mu(\{1,2\},\{3\}) =$$

$$= \mu(\{1,3\},\phi) = \mu(\{1,3\},\{1\}) = \mu(\{1,3\},\{2\}) = \mu(\{1,3\},\{3\}) = \mu(\{1,3\},\{1,2\}) =$$

$$= \mu(\{2,3\},\phi) = \mu(\{2,3\},\{1\}) = \mu(\{2,3\},\{2\}) = \mu(\{2,3\},\{3\}) = \mu(\{2,3\},\{1,2\}) =$$

$$\mu(\{2,3\},\{1,3\}) = \mu(\{1,2,3\},\phi) = \mu(\{1,2,3\},\{1\}) = \mu(\{1,2,3\},\{2\}) = \mu(\{1,2,3\},\{3\}) =$$

$$= \mu(\{1,2,3\},\{1,2\}) = \mu(\{1,2,3\},\{1,3\}) = \mu(\{1,2,3\},\{2,3\}) = 0$$

$$\mu(\phi,\{1\}) = -\mu(\phi,\phi) = -1;$$

$$\mu(\phi,\{2\}) = -\mu(\phi,\phi) = -1;$$

$$\mu(\phi,\{3\}) = -\mu(\phi,\phi) = -1;$$

$$\mu(\phi,\{1,2\}) = -\mu(\phi,\phi) - \mu(\phi,\{1\}) - \mu(\phi,\{2\}) = -1+1+1 = 1;$$

$$\mu(\phi,\{1,3\}) = -\mu(\phi,\phi) - \mu(\phi,\{1\}) - \mu(\phi,\{3\}) = -1+1+1 = 1;$$

$$\mu(\phi,\{2,3\}) = -\mu(\phi,\phi) - \mu(\phi,\{2\}) - \mu(\phi,\{3\}) = -1+1+1 = 1;$$

$$\mu(\phi,\{1,2,3\}) = -\mu(\phi,\phi) - \mu(\phi,\{1\}) - \mu(\phi,\{2\}) - \mu(\phi,\{3\}) - \mu(\phi,\{1,2\}) - \mu(\phi,\{1,3\})$$

$$-\mu(\phi,\{2,3\}) = -1+1+1+1-1-1-1 = -1;$$

$$\mu(\{1\},\{2\}) = \mu(\{1\},\{3\}) = \mu(\{1\},\{2,3\}) = \mu(\{2\},\{3\}) = \mu(\{2\},\{1,3\}) = \mu(\{3\},\{1,2\})$$
$$= \mu(\{1,2\},\{1,3\}) = \mu(\{1,2\},\{2,3\}) = \mu(\{1,3\},\{2,3\}) = 0;$$

$$\mu(\{1\},\{1,2\}) = -\mu(\{1\},\{1\}) = -1;$$

$$\mu(\{1\},\{1,3\}) = -\mu(\{1\},\{1\}) = -1;$$

$$\mu(\{1\},\{1,2,3\}) = -\mu(\{1\},\{1\}) - \mu(\{1\},\{1,2\}) - \mu(\{1\},\{1,3\}) = -1+1+1 = 1;$$

$$\mu(\{2\},\{1,2\}) = -\mu(\{2\},\{2\}) = -1;$$

$$\mu(\{2\},\{2,3\}) = -\mu(\{2\},\{2\}) = -1;$$

$$\mu(\{2\},\{1,2,3\}) = -\mu(\{2\},\{2\}) - \mu(\{2\},\{1,2\}) - \mu(\{2\},\{2,3\}) = -1+1+1 = 1;$$

$$\mu(\{3\},\{1,3\}) = -\mu(\{3\},\{3\}) = -1;$$

$$\mu(\{3\},\{2,3\}) = -\mu(\{3\},\{3\}) = -1;$$

$$\mu(\{3\},\{1,2,3\}) = -\mu(\{3\},\{3\}) - \mu(\{3\},\{1,3\}) - \mu(\{3\},\{2,3\}) = -1+1+1 = 1;$$

$$\mu(\{1,2\},\{1,2,3\}) = -\mu(\{1,2\},\{1,2\}) = -1;$$

$$\mu(\{1,3\},\{1,2,3\}) = -\mu(\{1,3\},\{1,3\}) = -1;$$

$$\mu(\{2,3\},\{1,2,3\}) = -\mu(\{2,3\},\{2,3\}) = -1.$$

Example 5. For the poset $P = \{1,2,3,4,5\}$ with the usual order of the positive integers we have $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, and in generally for a fixed positive integer $n \in \mathbf{N}$ and a poset $P = \{1,2,3,...,n\}$ with the usual order we obtain $A = \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix} \in M_n(\mathbf{R})$ and $A^{-1} = \begin{pmatrix} 1 & -1 & 0 & ... & 0 \\ 0 & 1 & -1 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$.

5.2. Classical Möbius inversion [8].

For functions f,g defined on the set $\mathbf{N}$ of positive integers we have:

$f(n) = \sum_{d/n} g(d) \Leftrightarrow g(n) = \sum_{d/n} \mu\left(\frac{n}{d}\right) f(d)$, for all $n \in \mathbf{N}$.

5.3. If we consider the set $P = \{1 < 2 < 3 < ... < n\}$ we have that the Möbius function associated to P is:

$$\mu(i,j) = \begin{cases} 1 & if\ \ i = j \\ -1 & if\ \ j = i+1 \\ 0 & else \end{cases}.$$

Then, for any functions $f, g, h : P \to G$, G is an commutative group, we have the relations:

$$g(m) = \sum_{i=m}^{n} f(i) \Leftrightarrow f(m) = g(m) - g(m+1)$$

$$h(m) = \sum_{i=0}^{m} f(i) \Leftrightarrow f(m) = h(m) - h(m-1)$$

And this is an application to the inversion formula of Möbius.

5.4. Let X be a finite set and $P = \mathrm{P}(X)$ with the relation given by inclusion. We have that the Möbius function of $\mathrm{P}(X)$ is:

$\mu(A,B) = (-1)^{|B-A|}$ for $A \subset B$.

Then, for any functions f, g, h : P $(P) \to G$, G is an commutative group, we have the relations:

$$g(A) = \sum_{A \subset B} f(B) \Leftrightarrow f(A) = \sum (-1)^{|B-A|} g(B)$$

$$h(A) = \sum_{B \subset A} f(B) \Leftrightarrow f(A) = \sum (-1)^{|A-B|} h(B).$$

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