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J.P.KING VERSION OF SCHURER OPERATOR

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Abstract: *In this paper we construct a Schurer type operator following a J.P.King model.*

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1 Introduction

Most of linear and positive operators on $C[a,b]$ preserve e_0 and e_1 :

$$\begin{aligned} L_n(e_0)(x) &= e_0(x) \\ L_n(e_1)(x) &= e_1(x) \end{aligned}$$

for each $n = 0, 1, 2, \ldots$ and $x \in [a,b]$.

J.P. King defined in [3] an interesting class of operators which preserve e_2. Let $(s_n(x))_{n\in\mathbb{N}}$ be a sequence of continuous functions on $[0,1]$ so that $0 \le s_n(x) \le 1$. For any $f \in C[0,1]$ and $x \in [0,1]$ let $V_n : C[0,1] \to C[0,1]$ be defined by

$$(V_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} s_n^k(x)(1 - s_n(x))^{n-k} f\left(\frac{k}{n}\right). \tag{1}$$

For $s_n(x) = x$, $n \in \mathbb{N}$ operators V_n become Bernstein operators. The values of the operators V_n on test functions $e_j = x^j$, $j = 0, 1, 2$ are given by

$$(V_n e_0)(x) = 1$$

$$(V_n e_1)(x) = s_n(x)$$

$$(V_n e_2)(x) = \frac{1}{n} s_n(x) + \frac{n-1}{n} s_n^2(x).$$

Using Bohman-Korovkin theorem ([1], [4]) it follows immediately that $\lim\limits_{n\to\infty} V_n f = f$ uniformly on $[0,1]$ if and only if $\lim\limits_{n\to\infty} s_n(x) = x$ uniformly on $[0,1]$.

In order to preserve e_2, the s_n sequence has to be as it follows:

$$\begin{cases} s_1(x) = x^2 \\ s_n(x) = -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, n = 2, 3, \ldots \end{cases}$$

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2 Main results

Let p be a non-negative integer. In 1962, F. Schurer [5] introduced the operators $\tilde{B}_{n,p}$: $C[0,1+p] \to C[0,1]$, defined for any $f \in C[0,1+p]$, $n \in \mathbb{N}$ and $x \in [0,1+p]$ by

$$\left(\tilde{B}_{n,p}f\right)(x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f\left(\frac{k}{n}\right). \tag{2}$$

For $p = 0$ they become the Bernstein operators. We recall the values of the Schurer operators on the test functions:

$$\left(\tilde{B}_{n,p}e_0\right)(x) = 1 \tag{3}$$

$$\left(\tilde{B}_{n,p}e_1\right)(x) = \left(1+\frac{p}{n}\right)x \tag{4}$$

$$\left(\tilde{B}_{n,p}e_2\right)(x) = \frac{n+p}{n^2}\left((n+p)x^2 + x(1-x)\right). \tag{5}$$

Let $(r_n(x))_{n\in\mathbb{N}}$ be a sequence of continuous functions on $[0,1+p]$ so that $0 \le r_n(x) \le 1+p$. We define now the operators $\tilde{V}_{n,p} : C[0,1+p] \to C[0,1]$ by

$$\left(\tilde{V}_{n,p}f\right)(x) = \sum_{k=0}^{n+p} \binom{n+p}{k} r_n^k(x)(1-r_n(x))^{n+p-k} f\left(\frac{k}{n}\right), \tag{6}$$

for any function $f \in C[0,1+p]$ and $x \in [0,1+p]$. It's obvious that for $r_n(x) = x$, $n \in \mathbb{N}$ the Schurer operators are obtained. The $\tilde{V}_{n,p}$ operators are linear and positive.

Teorema 1. *The operators $\tilde{V}_{n,p}$ defined above have the following properties:*

1. $\left(\tilde{V}_{n,p}e_0\right)(x) = 1$
2. $\left(\tilde{V}_{n,p}e_1\right)(x) = \frac{n+p}{n} r_n(x)$
3. $\left(\tilde{V}_{n,p}e_2\right)(x) = \frac{n+p}{n^2} r_n(x)\left((n+p-1)r_n(x)+1\right)$

Proof. 1. We have:

$$\left(\tilde{V}_{n,p}e_0\right)(x) = \sum_{k=0}^{n+p} \binom{n+p}{k} r_n^k(x)(1-r_n(x))^{n+p-k} = 1$$

2. The value on the test function e_1 can be written as it follows:

$$\left(\tilde{V}_{n,p}e_1\right)(x) = \sum_{k=0}^{n+p} \binom{n+p}{k} r_n^k(x)(1-r_n(x))^{n+p-k}\frac{k}{n} =$$

$$= \frac{n+p}{n}\sum_{k=1}^{n+p} \binom{n+p}{k} r_n^k(x)(1-r_n(x))^{n+p-k}\frac{k}{n+p}$$

and because $\binom{n+p}{k}\frac{k}{n+p} = \binom{n+p-1}{k-1}$ for any $k = \overline{1, n+p}$ we have

$$\left(\tilde{V}_{n,p}e_1\right)(x) = \frac{n+p}{n}\sum_{k=1}^{n+p} \binom{n+p-1}{k-1} r_n^k(x)(1-r_n(x))^{n+p-k} =$$

$$= \frac{n+p}{n}\sum_{k=0}^{n+p-1} \binom{n+p-1}{k} r_n^{k+1}(x)(1-r_n(x))^{n+p-k-1} =$$

$$= \frac{n+p}{n} r_n(x).$$

3. Similarily for the test function e_2 we obtain:

$$\left(\tilde{V}_{n,p}e_2\right)(x) = \sum_{k=0}^{n+p} \binom{n+p}{k} r_n^k(x)(1-r_n(x))^{n+p-k}\frac{k^2}{n^2} =$$

$$= \left(\frac{n+p}{n}\right)^2 \sum_{k=1}^{n+p} \binom{n+p}{k} r_n^k(x)(1-r_n(x))^{n+p-k}\frac{k^2}{(n+p)^2} =$$

$$= \left(\frac{n+p}{n}\right)^2 \sum_{k=1}^{n+p} \binom{n+p-1}{k-1} r_n^k(x)(1-r_n(x))^{n+p-k}\frac{k}{n+p} =$$

$$= \left(\frac{n+p}{n}\right)^2 \sum_{k=0}^{n+p-1} \binom{n+p-1}{k} r_n^{k+1}(x)(1-r_n(x))^{n+p-k-1}\frac{k+1}{n+p} =$$

$$= \left(\frac{n+p}{n}\right)^2 r_n(x)\left(\sum_{k=0}^{n+p-1} \binom{n+p-1}{k} r_n^k(x)(1-r_n(x))^{n+p-1-k}\frac{k}{n+p} + \frac{1}{n+p}\right) =$$

$$= \left(\frac{n+p}{n}\right)^2 r_n(x)\left(\frac{n+p-1}{n+p}\sum_{k=1}^{n+p-1} \binom{n+p-1}{k} r_n^k(x)(1-r_n(x))^{n+p-1-k}\frac{k}{n+p-1} + \frac{1}{n+p}\right) =$$

$$= \left(\frac{n+p}{n}\right)^2 r_n(x)\left(\frac{n+p-1}{n+p}\sum_{k=1}^{n+p-1} \binom{n+p-2}{k-1} r_n^k(x)(1-r_n(x))^{n+p-1-k} + \frac{1}{n+p}\right) =$$

$$= \left(\frac{n+p}{n}\right)^2 r_n(x)\left(\frac{n+p-1}{n+p}\sum_{k=0}^{n+p-2} \binom{n+p-2}{k} r_n^{k+1}(x)(1-r_n(x))^{n+p-2-k} + \frac{1}{n+p}\right) =$$

$$= \left(\frac{n+p}{n}\right)^2 r_n(x)\left(\frac{n+p-1}{n+p}r_n(x)\sum_{k=0}^{n+p-2} \binom{n+p-2}{k} r_n^k(x)(1-r_n(x))^{n+p-2-k} + \frac{1}{n+p}\right) =$$

$$= \left(\frac{n+p}{n}\right)^2 r_n(x)\left(\frac{n+p-1}{n+p}r_n(x) + \frac{1}{n+p}\right) =$$

$$= \frac{n+p}{n^2} r_n(x)\left((n+p-1)r_n(x)+1\right)$$

□

Let's see what form $r_n(x)$ should take in order to have $\tilde{V}_{n,p}e_2 = e_2$, that is

$$\frac{n+p}{n^2} r_n(x)\left((n+p-1)r_n(x)+1\right) = x^2$$

or

$$(n+p)(n+p-1)r_n^2(x) + (n+p)r_n(x) - n^2x^2 = 0.$$

If we denote

$$\begin{aligned} a &= (n+p)(n+p-1) \\ b &= n+p \\ c &= -n^2x^2 \end{aligned}$$

then the discriminant of the second grade equation is given by

$$\Delta = (n+p)^2 + 4n^2x^2(n+p)(n+p-1) \geq 0$$

for any $x \in [0, 1+p]$. For $n+p \neq 1$ the solutions of the equation are

$$(r_n(x))_{1,2} = \frac{-(n+p) \pm \sqrt{(n+p)^2 + 4n^2x^2(n+p)(n+p-1)}}{2(n+p)(n+p-1)}.$$

We choose

$$r_n^*(x) = \frac{-(n+p) + \sqrt{(n+p)^2 + 4n^2x^2(n+p)(n+p-1)}}{2(n+p)(n+p-1)}, \quad n > 1 \tag{7}$$

and

$$r_1^*(x) = x^2. \tag{8}$$

Lema 2. *For any $x \in \left[0, 1+\frac{p}{n}\right]$ the following inequality holds $0 \leq r_n^*(x) \leq 1$.*

Proof. Because $r_n^*(x) = \frac{-b+\sqrt{b^2-4ac}}{2a}$ the inequality $0 \leq r_n^*(x) \leq 1$ becomes

$$0 \leq \frac{-b + \sqrt{b^2 - 4ac}}{2a} \leq 1$$

Since $a > 0$ we get

$$0 \leq -b + \sqrt{b^2 - 4ac} \leq 2a$$
$$0 \leq b \leq \sqrt{b^2 - 4ac} \leq 2a + b$$

which leads to

$$b^2 \leq b^2 - 4ac \leq 4a^2 + 4ab + b^2$$
$$0 \leq -ac \leq a^2 + ab.$$

It results that we have to find $x \in [0, 1+p]$ such that

$$\begin{cases} c \leq 0 \\ a+b+c \geq 0 \end{cases}.$$

The first condition is met for any $x \in [0, 1+p]$. Replacing a, b, c *in* the second inequality becomes

$$(n+p)(n+p-1) + (n+p) - n^2x^2 \geq 0$$

or

$$x^2 \leq \left(\frac{n+p}{n}\right)^2,$$

therefore $x \in \left[-\frac{n+p}{n}, \frac{n+p}{n}\right]$ which eventually gives us that $x \in \left[0, \frac{n+p}{n}\right]$. □

Remark 3. *If we denote $I_n = \left[0, 1+\frac{p}{n}\right]$ from the inequality*

$$\frac{p}{n+1} \leq \frac{p}{n}$$

it follows that $I_n \subset I_{n+1}$, $n \in \mathbb{N}$; moreover for $n \to \infty$ the interval I_n becomes $[0, 1]$.

One can notice that $\lim_{n\to\infty} r_n^*(x) = x$, so we have the following

Teorema 4. *The operators $\tilde{V}_{n,p}$ given by (6) with the sequence $(r_n^*(x))_{n\in\mathbb{N}}$ defined by (7), (8) have the following properties:*

1. *they are linear and positive on $C[0,1+p]$*
2. $\left(\tilde{V}_{n,p}e_2\right)(x) = e_2(x)$, $n \in \mathbb{N}^*$ *for any* $x \in \left[0, 1+\frac{p}{n}\right]$
3. $\lim\limits_{n\to\infty}\left(\tilde{V}_{n,p}f\right)(x) = f(x)$ *for any* $f \in C[0,1+p]$, $x \in \left[0,1+\frac{p}{n}\right]$.

If L is a linear and positive operator on $C[a,b]$, then for any continuous function $f \in C[a,b]$ and $x \in [a,b]$ we have the evaluation (see [2], pg. 30)

$$|(Lf)(x) - f(x)| \leq |f(x)|\,|(Le_0)(x) - 1| + \left((Le_0)(x) + \frac{(L\varphi_x)(x)}{\delta}\right)\omega(f,\delta) \leq$$

$$\leq |f(x)|\,|(Le_0)(x) - 1| + \left((Le_0)(x) + \frac{\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}}{\delta}\right)\omega(f,\delta) \tag{9}$$

$(\forall)\, x \in I$, $(\forall)\, \delta > 0$, where $\varphi_x = e_1 - xe_0$

If the operator L satisfies the conditions $Le_0 = e_0$ şi $Le_2 = e_2$ then the evaluation (9) can be written as:

$$|(Lf)(x) - f(x)| \leq \left(1 + \frac{\sqrt{(L\varphi_x^2)(x)}}{\delta}\right)\omega(f,\delta)$$

and since

$$(L\varphi_x^2)(x) = L\left((e_1 - xe_0)^2, x\right) = (Le_2)(x) - 2x\,(Le_1)(x) + x^2\,(Le_0)(x) =$$

$$= 2x^2 - 2x\,(Le_1)(x) = 2x\,(x - (Le_1)(x)), \tag{10}$$

we can also write that

$$|(Lf)(x) - f(x)| \leq \left(1 + \frac{\sqrt{2x\,(x - (Le_1)(x))}}{\delta}\right)\omega(f,\delta)$$

for any $f \in C[a,b]$ and $x \in [a,b]$.

Since the operator L is positive and $\varphi_x^2 \geq 0$ we get that $L\varphi_x^2 \geq 0$ which is equivalent with $2x\,(x - (Le_1)(x))$. It follows that for any $x \in [a,b], a \geq 0$ the inequality

$$(Le_1)(x) \leq x.$$

holds true.

Taking $[a,b] = I_n$ and $L = \tilde{V}_{n,p}$ as a particular case we obtain:

Lema 5. *For any $x \in I_n$ if $r_n(x) = r_n^*(x)$ we have*

$$\left(\tilde{V}_{n,p}e_1\right)(x) \leq x.$$

We got that for any $x \in I_n$ we have $\left(\tilde{V}_{n,p}e_0\right)(x) = e_0(x)$, $\left(\tilde{V}_{n,p}e_2\right)(x) = e_2(x)$ şi $\left(\tilde{V}_{n,p}e_1\right)(x) \leq x$; therefore the following evaluation stands:

$$\left|(\tilde{V}_{n,p}f)(x) - f(x)\right| \leq \left(1 + \frac{\sqrt{2x\left(x - \left(\tilde{V}_{n,p}e_1\right)(x)\right)}}{\delta}\right)\omega(f,\delta).$$

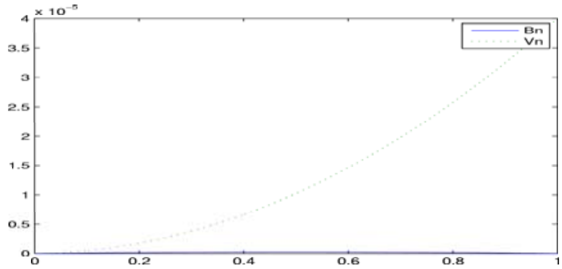
Replacing the values of $\tilde{V}_{n,p}$ on test functions in relation (10) we get the expression of the second order moment:

$$\left(\tilde{V}_{n,p}\varphi_x^2\right)(x) = 2x\left(x - \frac{n}{n+p}r_n^*(x)\right) \tag{11}$$

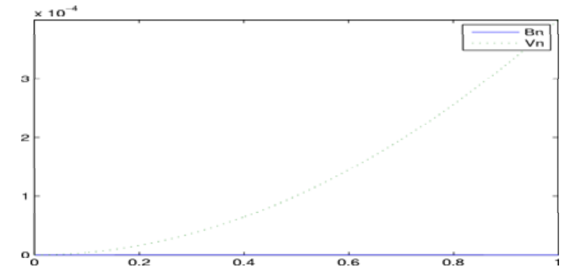
Using (3)-(5) in relation (10) the second order moment of Schurer operators is given by:

$$\left(\tilde{B}_{n,p}\varphi_x^2\right)(x) = \frac{1}{n^2}\left(p^2x^2 + (n+p)x(1-x)\right) \tag{12}$$

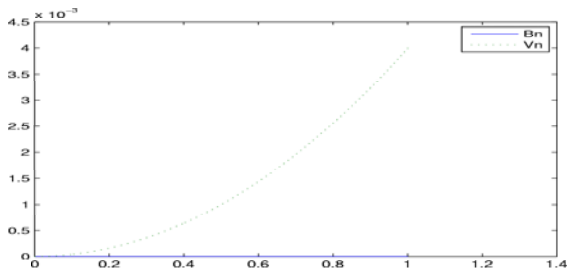
We present the graphics of the two moments for some particular cases



$n = 1.000.000;\ p = 10;$



$n = 1.000.000;\ p = 100;$



$n = 1.000.000;\ p = 1000;$

In these graphics one can see that

$$(\tilde{B}_{n,p}\varphi_x^2)(x) \leq (\tilde{V}_{n,p}\varphi_x^2)(x) \tag{13}$$

and $\tilde{V}_{n,p}\varphi_x^2 = O\left(\frac{p}{n}\right)$.

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