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SOME REMARKS ON THE NUMERICAL SERIES

ANDREI VERNESCU
Valahia University of Târgovişte, Bd. Unirii 18, 130082 Târgovişte, Romania, e-mail: *avernescu@clicknet.ro*

Abstract: *In the first part we present the obtaining of some numerical series. In the second part we do some remarks concerning the subseries of the harmonic series.*

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1 Obtaining the sum of certain series

In many cases, the summation of a nontrivial series is a very interesting process that can involve various techniques: the use of the power series, of the integrals in the real or complex domain, of the Fourier series and other.

We give here some examples of our works.

1. First consider the asymptotic representations of the regular lacunary harmonic series, namely let $r \in \mathbb{N}^*$, $r > 1$ be and $\alpha \in \mathbb{N}^*$, $1 \le \alpha \le r$. Consider the sums of [2]

$$H_{n,\alpha}^{(r)} := \frac{1}{\alpha} + \frac{1}{\alpha+r} + \frac{1}{\alpha+2r} + \ldots + \frac{1}{\alpha+(n-1)r} \tag{1.1}$$

We will use, for the harmonic sum $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, the asymptotic expression $H_n = \ln n + \gamma + \varepsilon_n$, where γ is the celebrated constant of Euler and $\varepsilon_n \to 0$.

To obtain the asymptotic representation of $H_{n,\alpha}^{(r)}$, consider the generating function $G_n^{[r]} : \mathbb{C} \to \mathbb{C}$ given by

$$G_n^{[r]}(u) = \frac{u}{1} + \frac{u^2}{2} + \frac{u^3}{3} + \ldots + \frac{u^{rn}}{rn}. \tag{1.2}$$

In [2] we have expressed the sum $H_{n,\alpha}^{(r)}$ as a linear combination of H_{rn} and the value of $G_n^{[r]}$ in the points ω_l ($\omega^r = 1$), $l = 1, 2, \ldots, r-1$ obtaining that

$$H_{n,\alpha}^{(r)} = \frac{1}{r} \sum_{l=0}^{r-1} \left(\omega_l^{\alpha}\right)^{-1} G_n^{[r]}\left(\omega_l\right), \tag{1.3}$$

$$H_{n,\alpha}^{(r)} = \frac{1}{r} \left(H_{rn} + \sum_{l=0}^{r-1} \left(\omega_l^{\alpha}\right)^{-1} G_n^{[r]}\left(\omega_l\right) \right). \tag{1.4}$$

But

$$G_n^{[r]}(u) = \int_0^1 \frac{1-z^{rn}}{1-z}\, \mathrm{d}z, \tag{1.5}$$

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that gives

$$G_n^{[r]}(u) = -\log(1-u) - I_n^r(u), \tag{1.6}$$

where log is the complex logarithm (an unspecified but fixed branch of it) and

$$I_n(u) = \int_0^u \frac{z^{rn}}{1-z}\,\mathrm{d}z \quad (\text{with } u \in \mathbb{C}\setminus[1,\infty)). \tag{1.7}$$

All these give the asymptotic representation

$$H_{n,\alpha}^{(r)} = \frac{1}{r}\left[\ln n + \ln r + \gamma - \right.$$
$$\left. -\sum_{l=1}^{r-1}\left(\cos\frac{2\alpha l\pi}{x}\cdot\left(2\sin\frac{l\pi}{r}\right) + \frac{(2l-r)\pi}{2}\cdot\sin\frac{2l\pi}{r}\right)\right] + o(1). \tag{1.8}$$

Some particular cases are concludent.

(a) For $r = 2$, we obtain

$$\begin{cases} H_{n,1}^{(2)} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} = \frac{1}{2}\ln n + \frac{1}{2}(\gamma + 2\ln 2) + o(1) \\ H_{n,2}^{(2)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} = \frac{1}{2}\ln 2 + \frac{1}{2}\gamma + o(1) \end{cases} \tag{1.9}$$

(b) For $r = 3$, we obtain

$$\begin{cases} H_{n,1}^{(3)} = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n-2} = \\ \quad = \frac{1}{3}\ln n + \frac{1}{3}\left(\gamma + \frac{3}{2}\ln 3 + \frac{\pi\sqrt{3}}{6}\right) + o(1); \\ H_{n,2}^{(3)} = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3n-1} = \\ \quad = \frac{1}{3}\ln n + \frac{1}{3}\left(\gamma + \frac{3}{2}\ln 3 - \frac{\pi\sqrt{3}}{6}\right) + o(1); \\ H_{n,3}^{(3)} = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \ldots + \frac{1}{3n} = \frac{1}{3}\ln n + \frac{1}{3}\gamma + + o(1). \end{cases} \tag{1.10}$$

The logarithm and the constant of Euler were "brotherly" divided by 3 and the terms containing π were apparead because of the complex logarithm.

(c) For $r = 4$, we have

$$\begin{cases} H_{n,1}^{(4)} = 1 + \frac{1}{5} + \frac{1}{9} + \ldots + \frac{1}{4n-3} = \\ \quad = \frac{1}{4}\ln n + \frac{1}{4}\left(\gamma + 3\ln 2 + \frac{\pi}{2}\right) + o(1); \\ H_{n,2}^{(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{1}{4n-2} = \\ \quad = \frac{1}{4}\ln n + \frac{1}{4}(\gamma + 2\ln 2) + o(1); \\ H_{n,3}^{(4)} = \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{4n-1} = \\ \quad = \frac{1}{4}\ln n + \frac{1}{4}\left(\gamma + 3\ln 2 - \frac{\pi}{2}\right) + o(1); \\ H_{n,4}^{(4)} = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \ldots + \frac{1}{4n} = \frac{1}{4}\ln n + \frac{1}{4}\gamma + o(1). \end{cases} \tag{1.11}$$

A similar comment is valid.

2. Of course, $H_{n,\alpha}^{(r)}$ diverges, but the sequence of general term

$$x_{n,\alpha}^{(r)} = H_{n,\alpha}^{(r)} - \frac{1}{r}\ln n \tag{2.1}$$

converges; let $x_{\alpha}^{(r)}$ be its limit. In [4] we have characterized the speed of convergence of $\left(x_{n,\alpha}^{(r)}\right)_n$, namely

$$\begin{cases} \lim\limits_{n\to\infty} n\left(x_{\alpha}^{(r)} - x_{n,\alpha}^{(r)}\right) = \dfrac{r-2\alpha}{2r^2}, & \text{if} \quad \alpha < \dfrac{r}{2} \\ \lim\limits_{n\to\infty} n\left(x_{n,\alpha}^{(r)} - x_{\alpha}^{(r)}\right) = \dfrac{r-2\alpha}{2r^2}, & \text{if} \quad \alpha \geq \dfrac{r}{2}. \end{cases} \tag{2.2}$$

We also have obtained a new accelerated convergence to the constant γ of Euler, via $\ln 2$, namely

$$\begin{aligned} 2\left(\ln 2 - \frac{1}{48n^2}\right) < 2\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right) - \ln n - \gamma < \\ < 2\left(\ln 2 + \frac{1}{48(n-1)^2}\right). \end{aligned} \tag{2.3}$$

3. To obtain certain series, the integrals are useful. So, let for $q > 0$

$$S(q) = 1 - \frac{1}{q\cdot 1+1} + \frac{1}{q\cdot 2+1} - \ldots + \frac{(-1)^n}{q\cdot n+1} + \ldots. \tag{3.1}$$

be. Denote by $S_n(q)$ the n-th sum of $S(q)$, namely

$$S_n(q) = 1 - \frac{1}{q\cdot 1+1} + \frac{1}{q\cdot 2+1} - \ldots + \frac{(-1)^n}{q\cdot n+1}. \tag{3.2}$$

In [1] we have obtained that

$$S(q) = \int_0^1 \frac{\mathrm{d}x}{1+x^q} \tag{3.3}$$

and further

$$S(q) = \frac{1}{q}\beta\left(\frac{1}{q}\right), \tag{3.4}$$

where β is a classic transcendental function, namely

$$\beta(x) \overset{\text{def}}{=} \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right], \tag{3.5}$$

with

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The first particular examples for $q \in \mathbb{N}$, $q > 1$, are well-known:

$$\begin{cases} 1 - \dfrac{1}{2} + \dfrac{1}{3} - \ldots + \dfrac{(-1)^n}{n+1} + \ldots = \displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x} = \ln 2 \\ 1 - \dfrac{1}{3} + \dfrac{1}{5} - \ldots + \dfrac{(-1)^n}{2n+1} + \ldots = \displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{4} \quad \text{(Leibniz)} \\ 1 - \dfrac{1}{4} + \dfrac{1}{7} - \ldots + \dfrac{(-1)^n}{3n+1} + \ldots = \displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x^3} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2\right) \\ 1 - \dfrac{1}{5} + \dfrac{1}{9} - \ldots + \dfrac{(-1)^n}{4n+1} + \ldots = \displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x^4} = \frac{1}{4\sqrt{2}}\left(2\ln\left(1+\sqrt{2}\right) + \pi\right). \end{cases} \tag{3.6}$$

4. The theory of power series also can give a strong method to obtain the sums of the series. Let

$$\Omega_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \tag{4.1}$$

be. In [5], using the well-known formula

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} \Omega_n x^{2n}, \quad x \in (-1,1) \tag{4.2}$$

that gives

$$\frac{1}{x\sqrt{1-x^2}} = \frac{1}{x} + \sum_{n=1}^{\infty} \Omega_n x^{2n-1} \tag{4.3}$$

and our original one

$$\frac{1}{x\sqrt{1-x^2}} = \frac{1}{x} - \frac{\left(1+\sqrt{1-x^2}\right)'}{1+\sqrt{1-x^2}}, \tag{4.4}$$

we has obtained by an integration on a compact $[a,b] \subset (0,1)$ and doing $a \searrow 0$ and $b \nearrow 0$ that

$$\sum_{n=1}^{\infty} \Omega_n \frac{1}{n} = \ln 4. \tag{4.5}$$

Also, starting from the "dual" formula

$$\frac{1}{\sqrt{1+x^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \Omega_n x^{2n}, \quad x \in (-1,1) \tag{4.2$'$}$$

and

$$\frac{1}{x\sqrt{1+x^2}} = \frac{1}{x} - \frac{\left(1+\sqrt{1+x^2}\right))'}{1+\sqrt{1+x^2}} \tag{4.4$'$}$$

we have obtained the alternate series corresponding to (4.5), namely

$$\sum_{n=1}^{\infty} (-1)^n \Omega_n \frac{1}{n} = 2 \ln \frac{2}{1+\sqrt{2}}. \tag{4.5$'$}$$

Also, we have obtained

$$\sum_{n=1}^{\infty} \Omega_n \frac{1}{2n+1} = \frac{\pi}{2} - 1 \tag{4.6}$$

$$\sum_{n=1}^{\infty} (-1)^n \Omega_n \frac{1}{2n+1} = \log(1+\sqrt{2}) - 1 \tag{4.7}$$

$$\sum_{n=1}^{\infty} \Omega_n \frac{1}{n+1} = 1 \tag{4.8}$$

$$\sum_{n=1}^{\infty} (-1)^n \Omega_n \frac{1}{n+1} = 2\sqrt{2} - 3. \tag{4.9}$$

Our main result of [5] is

$$\sum_{n=1}^{\infty} \Omega_n \frac{1}{n+p} = 2^{2p-1} \frac{\Gamma^2(p)}{\Gamma(2p)} - \frac{1}{p}. \tag{4.10}$$

2 Some remarks concerning the subseries of the harmonic series

5. The harmonic series $\sum_{n=1}^{\infty} 1/n$ is divergent, but some of it subseries are convergent. A subseries of the harmonic series is a series of the form $\sum_{n=1}^{\infty} 1/k_n$, where $(k_n)_n$ is a strictly increasing sequence of natural numbers.

Some examples are remarkable, as the following:

(a) $\sum_{n=1}^{\infty} 1/q^n$, with $q \in \mathbb{N}^*$, $q > 1$; it is a geometric series that has as ratio the inverse of the natural integer q.

(b) $\sum_{n=1}^{\infty} 1/n^s$, with $s \in \mathbb{N}^*$, $s > 1$; it is a generalized harmonic series of exponent $s > 1$.

(c) $\sum_{n=0}^{\infty} 1/n!$; it is the exponential series; its sum is the famous number e.

(d) $\sum_{n=0}^{\infty} 1/a^{n!}$, with $a \in \mathbb{N}$, $a > 1$; it is a number of Liouville, of natural basis and it represents a transcendental number.

A converse affirmation of (b) can be easily established, using the so-called

Lemma of C. L. Siegel. If s is a real number such that 2^s, 3^s and 5^s are simultaneously natural numbers, then s is a natural number.

We have the following

Proposition 1. *If a generalized harmonic series $\sum_{n=1}^{\infty} 1/n^s$ is a subseries of the harmonic series, then s is a natural number.*

Proof. Indeed, because of the considered series it a subseries of the harmonic series, 2^s, 3^s and 5^s are natural numbers. By the lemma of Siegel, $s \in \mathbb{N}$.

6. Considering again a subseries of the harmonic series, that will be now denoted $\sum_{k=1}^{\infty} 1/n_k$, we remark that if the limit

$$\lambda \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{n_k}{k}$$

exists, it is finite and $\lambda \neq 0$, then the series is divergent.

The converse affirmation is not true; taking $n_k = [k \ln k]$, the corresponding series is divergent, but $\lambda = \infty$! We have moreover:

Proposition 2. *If the series $\sum_{k=1}^{\infty} 1/n_k$ is convergent and the precedent limit exists, then $\lambda = \infty$.*

Proposition 3. *Let $\sum 1/x_n$ a convergent subseries of the harmonic series. Then:*

(i) *The limit $\lambda = \lim_{n\to\infty} x_n/n$ exists.*

(ii) $\lambda = \infty$.

7. Considering the infinite, ordered sequence of the prime numbers $p_1 < p_2 < p_3 < \ldots < p_n \ldots$ ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$), it is well known that the series $\sum_{n=1}^{\infty} p_n$ is divergent.

In the oldest and well known proof of the infinity of the primes of Euclid, supposing that the set of all the primes is finite, namely

$$\mathcal{P} = \{p_1, p_2, \ldots, p_n\},$$

the number $q = p_1 p_2 \cdot \ldots p_n$ is sufficient to find a contradiction.

We can construct now, by analogy to the precedent construction (but not considering the hypotetic previous set $\mathcal{P}$) for any n the member

$$q_n = p_1 p_2 \cdot \ldots \cdot p_n + 1$$

where $p_1, p_2, \ldots, p_n$is the infinite sequence of the primes (so $q_1 = 3$, $q_2 = 7$, $q_3 = 31$ etc). We have

Proposition 4 *The series* $\sum_{n=1}^{\infty}$ *is convergent* (*see* [3]).

We also have by Bertrand's postulate

Proposition 5. *It exists a convergent subseries of the divergent series* $\sum_{n=1}^{\infty} 1/p_n$.

Proposition 6. *For any* $\varepsilon > 0$ *there are two convergent subseries* S' *and* S'' *of the divergent series* $\sum_{n=1}^{\infty} 1/p_n$, *such that* $S' > S''$ and

$$S' - S'' < \varepsilon$$

(also see [3]).

8. A result of Kempner is well-known: If in the harmonic series we eliminate of the terms $1/n$ such that n contains the digit 9, then the new obtained series is convergent.

The most recent result related to his series in the one of Bakir Farhi, that will appear in the issue of December 2008 of AMM, intitled "A curious result related to Kempner's series".

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