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LOCALIZATION OF POSITIVE SOLUTIONS OF SYSTEMS OF HAMMERSTEIN EQUATIONS

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Abstract: *We establish conditions which guarantee the localization of positive solutions of nonlinear integral equations of Hammerstein type.*

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1 Introduction

Many mathematical models arise to problem for which only nonnegative solutions make sense. Therefore, attention of the researchers is capture by the studies of existence of positive solutions of nonlinear equations. The compression-expansion theorems are an important tool in the study of the existence of positive solutions to several type of differential or integral equations. In the last years, the Krasnoselskii's fixed point theorem and its generalizations have been successfully applied to obtain existence results for nonnegative solutions of various type of problems [3, 4, 12, 14, 15].

Localization of solutions for nonlinear operator equations can be obtained by using variational methods, upper and lower solution method or existence results related to ordered Banach spaces. In this paper we seek positive solutions of the nonlinear integral equation

$$u(t) = \int_{\Omega} G(t,s) f(s,u(s))\, ds, \quad t \in \Omega, \tag{1}$$

where $\mathbb{R}$ is the set of real numbers, $\Omega \subset \mathbb{R}^n$ is a compact set, $n \geq 2$, $G : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times [0, R] \to \mathbb{R}$ with $R > 0$.

The main tool to our approach is a new variant of compression-expansion theorem due by R.W. Legget and L.R. Williams [9].

2 Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Let X be a Banach space endowed with the norm $|\cdot|$. A nonempty closed convex set $K \subset X$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in K, \lambda \geq 0$ implies $\lambda x \in K$;

(ii) $x \in K, -x \in K$ implies $x = 0$.

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We say that $\sigma : K \to [0,\infty)$ is a *nonnegative concave functional* over the cone K if σ is continuous and for every $x, y \in K$ and $\lambda \in [0,1]$ we have

$$\sigma(\lambda x + (1-\lambda) y) \geq \lambda \sigma(x) + (1-\lambda)\sigma(y).$$

Assume that $\sigma(x) \leq |x|$ for any $x \in K$. For $0 < a < b$ we consider the set

$$\Sigma(\sigma, a, b) = \{x \in K : \ a \leq \sigma(x) \ \text{and} \ |x| \leq b\}.$$

For $R > 0$ we denote by

$$K_R = \{x \in K; \ |x| \leq R\}$$

the closed ball of radius R from the cone K.

We say that the applications $T : K_R \to K$ is an *operator of Mönch type* if T is continuous and there exists $x_0 \in K$ such that for any countable set $C \subset K$, the inclusion $C \subset co\{\{x_0\} \cup T(C)\}$ implies C relatively compact. Examples of operators of Mönch type can be found in [5, 10]. The existence results for operator of Mönch type are presented in [4, 5, 6] and many others.

The main tool in our approach is the following theorem:

Theorem 1. *Assume that $T : K_R \to K$ is an operator of Mönch type and there are two numbers $\rho, r \in (0, R]$ with $\rho < r$ such that*

($\mathbf{W}_1$) *the set $\{x \in \Sigma(\sigma, \rho, r) : \ \sigma(x) > \rho\}$ is nonempty;*

($\mathbf{W}_2$) *$\sigma(T(x)) > \rho$ for any $x \in \Sigma(\sigma, \rho, r)$;*

($\mathbf{W}_3$) *$\sigma(T(x)) > \rho$ for any $x \in \Sigma(\sigma, \rho, R)$ with $|T(x)| > r$.*

Then T has at least one fixed point in $\Sigma(\sigma, \rho, R)$.

This result is an extension of the theorem obtained by R.W. Legget and L.R. Williams in [9] from compact operators to the operators of Mönch type.

Remark that ($\mathbf{W}_3$) is guarantee by one of the following conditions:

$$\sigma(T(x)) \geq \frac{\rho}{r}|T(x)| \ \text{for any } x \in \Sigma(\sigma, \rho, R) \tag{2}$$

or

$$|T(x)| - \sigma(T(x)) \leq r - \rho, \ \text{for any } x \in \Sigma(\sigma, \rho, R), \tag{3}$$

3 Main result

Let $\Omega_1 \subset \Omega$ be a closed subset of Ω of positive Lebesgue measure. Suppose that

$$|G| = \sup_{t\in\Omega} \int_{\Omega} G(t,s)\,ds < \infty,$$

$$\gamma_0 = \inf_{t\in\Omega_1} \int_{\Omega_1} G(t,s)\,ds > 0,$$

$$\delta = \sup_{\substack{t\in\Omega \\ \tau\in\Omega_1}} \int_{\Omega} |G(t,s) - G(\tau,s)|\,ds > 0.$$

($\mathbf{H}_1$) The maps $f : \Omega \times [0,R] \to \mathbb{R}_+$ and $G : \Omega \times \Omega \to \mathbb{R}$ are such that the operator $T : C(\Omega; \mathbb{R}_+) \to C(\Omega; \mathbb{R}_+)$ given by

$$T(u)(t) = \int_{\Omega} G(t,s) f(s, u(s))\,ds, \ \ t \in \Omega$$

is operator of Mönch type.

$(\mathbf{H}_2)$ There are $\rho, r \in (0, R)$ with $\rho < r$ and $\gamma \in (0, \gamma_0]$ such that

$$f(t,x) > \frac{\rho}{\gamma} \text{ for every } t \in \Omega_1 \text{ and } x \in [\rho, r];$$

$(\mathbf{H}_3)$ For every $t \in \Omega$ and $x \in [0, R]$ we have

$$f(t,x) < \frac{R}{|G|};$$

$(\mathbf{H}_4)$ $0 < \rho < r < R$

$$\frac{\delta}{|G|} < \frac{r-\rho}{R}.$$

Theorem 2. *Assume that* $(\mathbf{H}_1) - (\mathbf{H}_4)$ *are satisfied. Then, there is* $u \in C(\Omega; \mathbb{R}_+)$ *a positive solution of integral equation*

$$u(t) = \int_{\Omega} G(t,s) f(s, u(s))\, ds, \quad t \in \Omega. \tag{4}$$

Much more, we have the following localization

$$\begin{cases} \rho \leq \min\limits_{t \in \Omega_1} u(t) \\ \max\limits_{t \in \Omega} u(t) \leq R. \end{cases} \tag{5}$$

Proof. We will apply the Theorem 1. To do this, we consider the positive cone $K = C(\Omega, \mathbb{R}_+)$ of all continuous maps from Ω to $\mathbb{R}_+$ and the set

$$K_R = \left\{ u \in K : \ \max_{t \in \Omega} u(t) \leq R \right\}.$$

For $0 < a < b$ let

$$\Sigma(a,b) = \left\{ u \in K : \ a \leq \min_{t \in \Omega_1} u(t) \ \text{și} \ \max_{t \in \Omega} u(t) \leq b \right\}$$

be the set which contains the desired positive solutions. Remark that

$$\Sigma(a,b) = \Sigma(\sigma, a, b),$$

where $\sigma : K \to \mathbb{R}_+$ is given by $\sigma(u) = \min\limits_{t \in \Omega_1} |u(t)|$ for any $u \in C(\Omega)$. The set

$$\left\{ u \in \Sigma(\rho, r) : \rho < \min_{t \in \Omega_1} u(t) \right\}$$

is nonempty, so $(\mathbf{W}_1)$ is satisfied.

Let $u \in \Sigma(\rho, r)$, then $u \in C(\Omega; \mathbb{R}_+)$ with

$$\rho \leq \min_{t \in \Omega_1} u(t) \text{ and } \max_{t \in \Omega} u(t) \leq r.$$

So, for any $t \in \Omega_1$, we have

$$\rho \leq u(t) \leq r. \tag{6}$$

Hence, from $(\mathbf{H}_2)$, for every $s \in \Omega_1$ and $u \in \Sigma(\rho, r)$ we have

$$f(s, u(s)) > \frac{\rho}{\gamma}.$$

Then

$$\min_{t\in\Omega_1} T(u)(t) = \min_{t\in\Omega} \int_{\Omega_1} G(t,s) f(s,u(s))\, ds$$
$$\geq \min_{t\in\Omega_1} \int_{\Omega_1} G(t,s) f(s,u(s))\, ds$$
$$> \frac{\rho}{\gamma} \cdot \min_{t\in\Omega_1} \int_{\Omega_1} G(t,s)\, ds.$$

Since $\gamma \in (0, \gamma_0]$, we obtain

$$\min_{t\in\Omega_1} T(u)(t) > \frac{\rho}{\gamma_0} \min_{t\in\Omega_1} \int_{\Omega_1} G(t,s)\, ds = \rho$$

for any $u \in \Sigma(\rho, r)$. This guarantees that $(\mathbf{W}_2)$ holds.

Let $u \in \Sigma(\rho, R)$. Then $u \in C(\Omega; \mathbb{R}_+)$ with

$$\rho \leq \min_{t\in\Omega_1} u(t) \text{ and } \max_{t\in\Omega} u(t) \leq R.$$

Now $(\mathbf{H}_3)$ ensures that

$$f(s, u(s)) \leq \frac{R}{|G|} \text{ for any } s \in \Omega \text{ and } u \in \Sigma(\rho, R).$$

Let $t \in \Omega$ and $\tau \in \Omega_1$. Then

$$\int_{\Omega} G(t,s) f(s,u(s))\, ds - \int_{\Omega} G(\tau,s) f(s,u(s))\, ds \leq$$
$$\leq \int_{\Omega} |G(t,s) - G(\tau,s)|\, f(s,u(s))\, ds.$$
$$\leq \frac{R}{|G|} \int_{\Omega} |G(t,s) - G(\tau,s)|\, ds$$
$$\leq \delta \cdot \frac{R}{|G|}. \tag{7}$$

The hypothesis $(\mathbf{H}_4)$ and (7), imply that

$$\max_{t\in\Omega} \int_{\Omega} G(t,s) f(s,u(s))\, ds - \min_{\tau\in\Omega_1} \int_{\Omega} G(\tau,s) f(s,u(s))\, ds \leq r - \rho \tag{8}$$

holds for any $u \in \Sigma(\rho, R)$. The inequality (8) implies (3), therefore $(\mathbf{W}_3)$ is guarantees.

Now, we can apply Theorem 1 and we obtain that nonlinear integral equation (1) has at least one solution in $\Sigma(\rho, R)$. This is equivalent to localization (5).

$\square$

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