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ON THE EXISTENCE AND STABILITY OF PERIODIC SOLUTION FOR BAM NEURAL NETWORKS WITH DELAYS

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Abstract: *The purpose of this paper is to present some sufficient conditions ensuring existence, uniqueness and global stability of almost periodic solution of the BAM neural networks with variable coefficients and delays, by using the Banach fixed point theorem and constructing suitable Lyapunov function.*

Keywords: *BAM neural networks; Almost periodic solution; global stability*

1 Introduction

Neural networks have started to be used for the last years in many fields such as pattern recognition, automatic control, function approximation, vocal recognition, financial prediction, risking management and so on [4],[6]. This domain attracted many scholars, based on the stability of bidirectional associative memory neural networks with and without delays [1].

There have been obtained some sufficient conditions for globally asymptotic stability of delayed bidirectional associative memory networks (BAM). Moreover, authors research the periodic oscillatory solution of BAM networks with delays in the case of constant coefficients [2]. Also, authors in [7] investigated the existence and stability of positive periodic solution by using the Krasnoselskii fixed point theory. It is well known that studies on neural dynamical systems not only imply a discussion of stability properties, but also imply many dynamic behaviours such as periodic oscillatory behaviour, almost periodic oscillatory properties, chaos, and bifurcation. In applications, almost periodic oscillatory is more accordant with fact. To the best of our knowledge, few authors have considered almost periodic oscillatory solutions for BAM networks, most of them study the stability, periodic oscillation of BAM networks in the case of constant coefficients.

In this paper, we discuss almost periodic oscillatory solutions of BAM networks with variable coefficients and delays, present some simple sufficient conditions obtained in [3] by using the Banach fixed point theorem and constructing suitable Lyapunov function, ensuring the existence and global asymptotic stability of almost periodic solution.

In the end, we give an example to illustrate the feasibility of these results.

2 Preliminaries

Consider the BAM networks with variable coefficients

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i(t)\,x_i(t) + \sum_{j=1}^{p} p_{ji}(t)\,f_j\left(y_j\left(t-\tau_{ji}\right)\right) + I_i(t) \quad \text{(1a)}$$

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$$\frac{\mathrm{d}y_j}{\mathrm{d}t} = -b_j(t)\, y_j(t) + \sum_{i=1}^{n} q_{ij}(t)\, f_i\left(x_i\left(t-\sigma_{ij}\right)\right) + J_j(t) \quad (1b)$$

where $i = \overline{1,n}$, $j = \overline{1,p}$, $x_i(t)$ and $y_j(t)$ are the activations of the i^{th} and the j^{th} neurons, respectively $p_{ji}(t)$, $q_{ij}(t)$ are the connection weights at the time t, $I_i(t)$ and $J_i(t)$ denote the external inputs at time t. The numbers τ_{ji} and σ_{ij} are nonnegative constants, which correspond to the finite speed of the axonal signal transmission.

Throughout this paper, we always assume that $a_i(t)$, $b_j(t)$, $p_{ji}(t)$, $q_{ij}(t)$, $I_i(t)$, $J_j(t)$ are continuous almost periodic functions.

Moreover, $a_i(t)$, $b_j(t)$ are positive,

$$0 < \inf_{t\in\mathbb{R}} \{a_i(t)\} = a_i^-, \quad 0 < \inf_{t\in\mathbb{R}} \{b_j(t)\} = b_j^-,$$

$$p_{ji}^+ = \sup_{t\in\mathbb{R}} \{|p_{ji}(t)|\} < +\infty,\ q_{ij}^+ = \sup_{t\in\mathbb{R}} \{|q_{ij}(t)|\} < +\infty,$$

$$I_i^+ = \sup_{t\in\mathbb{R}} \{|I_i(t)|\} < +\infty,\ J_j^+ = \sup_{t\in\mathbb{R}} \{|J_j(t)|\} < +\infty.$$

Furthermore, the signal functions f_i possess the following properties:

(H1) f_i are bounded on $\mathbb{R}$, for all $i = \overline{1, \max\{n,p\}}$;

(H2) For $i = \overline{1, \max\{n,p\}}$, there exists a number $\mu_i > 0$ such that

$$|f_i(x) - f_i(y)| \le \mu_i |x-y|.$$

The initial conditions associated to (1a), (1b) are connected the form

$$\begin{cases} x_i(t) = \Phi_i(s),\, s\in[-\tau,0],\, \tau = \max\limits_{1\le i\le n}\max\limits_{1\le j\le p} \{\tau_{ji}\}, \\ y_j(t) = \Psi_i(s),\, s\in[-\sigma,0],\, \sigma = \max\limits_{1\le i\le p}\max\limits_{1\le j\le n} \{\sigma_{ji}\}, \end{cases} \quad (1)$$

where $\Phi_i(s)$, $\Psi_i(s)$ are continuous almost periodic functions on $\mathbb{R}$.

For any solution $z^*(t) \triangleq (x^*(t),\ y^*(t))^T$, i.e.

$$z^*(t) \triangleq \left(x_1^*(t), x_2^*(t), \ldots, x_n^*(t), y_1^*(t), y_2^*(t), \ldots, x_p^*(t)\right)^T$$

of system (1a),(1b) we define the norm

$$\begin{aligned} \left\|(\Phi,\Psi)^T - (x^*,y^*)^T\right\| &= \sup_{-\tau\le t\le 0}\left\{\sum_{i=1}^{n}\left(\Phi_i(t) - x_i^*(t)\right)^2\right\} \\ &+ \sup_{-\sigma\le t\le 0}\left\{\sum_{j=1}^{p}\left(\Psi_j(t) - y_j^*(t)\right)^2\right\}. \end{aligned}$$

3 Existence of almost periodic solution

For an arbitrary vector

$$z(t) = \left(x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_p(t)\right)^T$$

we define the norm

$$\|z(t)\| = \max_{i=1,n} |x_i(t)| + \max_{j=1,p} |y_j(t)|.$$

Denote

$$S^{n+p} = \left\{ z \,\middle|\, z = \left(\Phi_1, \Phi_2, \ldots, \Phi_n, \Psi_1, \Psi_2, \ldots, \Psi_p\right)^T \right\},$$

where Φ_i, Ψ_j are continuous almost periodic functions on $\mathbb{R}$, with $i=\overline{1,n}$, $j=\overline{1,p}$.

For any $z \in S^{n+p}$, we define

$$\|z\| = \sup_{t\in\mathbb{R}} \|z(t)\| = \sup_{t\in\mathbb{R}} \max_{i=1,n} |\Phi_i(t)| + \sup_{t\in\mathbb{R}} \max_{j=1,p} |\Psi_j(t)|,$$

the induced module. Then S^{n+p} is a Banach space.

Theorem 3.1. *[3].*

In addition to (H1) and (H2), suppose further that

(H3) $r = \max_{i=1,n} \left\{ \frac{\sum_{j=1}^{p} p_{ji}^{+}\mu_j}{a_i^{-}} \right\} + \max_{j=1,p} \left\{ \frac{\sum_{i=1}^{n} q_{ij}^{+}\mu_i}{b_j^{-}} \right\} < 1$

(H4)

$$M[a_i] = \lim_{T\to\infty} \int_t^{t+T} a_i(s)ds > 0$$

$$M[b_j] = \lim_{T\to\infty} \int_t^{t+T} b_j(s)ds > 0,$$

where $i=\overline{1,n}$ *and* $j=\overline{1,p}$.

Then there is a unique periodic solution of system (1a)+(1b) in the region $\|z-z_0\| \le \frac{rI}{1-r}$, *with* $I = \max_{i=1,n} \left\{ \frac{I_i^{+}}{a_i^{-}} \right\} + \max_{j=1,p} \left\{ \frac{J_j^{+}}{b_j^{-}} \right\}$ *and*

$$z_0 = (A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n),$$

where $A_i = \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} I_i(s)ds$, $B_j = \int_{-\infty}^{t} e^{-\int_s^t b_j(u)du} J_j(s)ds$ *for* $i=\overline{1,n}$ *and* $j=\overline{1,p}$.

Proof. For any $(\Phi,\Psi)^T = (\Phi_1, \Phi_2, \ldots, \Phi_n, \Psi_1, \Psi_2, \ldots, \Psi_p)^T \in S^{n+p}$, we consider the almost solution $z_{(\Phi,\Psi)^T}$ of nonlinear almost periodic differential equation

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i(t)\, x_i(t) + \sum_{j=1}^{p} p_{ji}(t)\, f_j(\Psi_j(t-\tau_{ji})) + I_i(t), \text{(2a)}$$

$$\frac{\mathrm{d}y_j}{\mathrm{d}t} = -b_j(t)\, y_j(t) + \sum_{i=1}^{n} q_{ij}(t)\, f_i(\Phi_i(t-\sigma_{ij})) + J_j(t), \text{(2b)}$$

For $i=\overline{1,n}$, we denote

$$C_i = \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} \left[\sum_{j=1}^{p} p_{ji}(s)\, f_j(\Psi_j(s-\tau_{ji})) + I_i(s) \right] ds$$

and for $j=\overline{1,p}$,

$$D_j = \int_{-\infty}^{t} e^{-\int_s^t b_j(u)du} \left[\sum_{i=1}^{n} q_{ji}(s)\, f_i(\Phi_i(s-\sigma_{ij})) + J_j(s) \right] ds$$

.

The solution $z_{(\Phi,\Psi)^T}$ of (1) can be expressed as following

$$z_{(\Phi,\Psi)^T} = (C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_p).$$

Now we define a mapping $T : S^{n+p} \to S^{n+p}$,

$$T(\Phi, \Psi)(t) = z_{(\Phi,\Psi)^T}.$$

Let $B^* = \left\{ z \,|\, z \in S^{n+p},\ \|z - z_0\| \le \frac{rI}{1-r} \right\}$ be the closed ball from S^{n+p} with radius $\frac{rI}{1-r}$. Then B* is a closed convex subset of S^{n+p}. According to the definition of the norm of Banach space S^{n+p}, we have

$$\begin{aligned} \|z_0\| &= \sup_{t\in\mathbb{R}} \max_{i=1,n} \left| \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} I_i(s) ds \right| \\ &+ \sup_{t\in\mathbb{R}} \max_{j=1,p} \left| \int_{-\infty}^{t} e^{-\int_s^t b_j(u)du} J_j(s) ds \right| \\ &\le \sup_{t\in\mathbb{R}} \max_{i=1,n} \int_{-\infty}^{t} |I_i(s)| e^{-\int_s^t a_i(u)du} ds \\ &+ \sup_{t\in\mathbb{R}} \max_{j=1,p} \int_{-\infty}^{t} |J_j(s)| e^{-\int_s^t b_j(u)du} ds \\ &\le \max_{i=1,n} \left\{ \frac{I_i^+}{a_i^-} \right\} + \max_{j=1,p} \left\{ \frac{J_j^+}{b_j^-} \right\} = I. \end{aligned}$$

Therefore, $\|z\| \le \|z - z_0\| + \|z_0\| = \frac{rI}{1-r} + I = \frac{I}{1-r}$. It could be proved that the mapping T is a self-mapping from B* to B*.

Then, we prove that the mapping T is a contraction mapping of B. In fact, in view of (H1) and (H2), for any z1; z2 2 B, where $z_1, z_2 \in B*$, where

$$z_1 = (\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_p)^T,$$

$$z_2 = (\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_p)^T,$$

we have

$$\begin{aligned} \|T(z_1) - T(z_2)\| &= \sup_{t\in\mathbb{R}} \max_{i=1,n} \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} \cdot \right.\right. \\ &\left.\left. \sum_{j=1}^{p} p_{ji}(s) \left(f_j(\xi_j(s - \tau_{ji})) - f_j(\varphi_j(s - \tau_{ji})) \right) ds \right| \right\} \\ &+ \sup_{t\in\mathbb{R}} \max_{j=1,p} \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t b_j(u)du} \right.\right. \\ &\left.\left. \cdot \sum_{i=1}^{n} q_{ij}(s) \left(f_i(\eta_i(s - \sigma_{ij})) - f_i(\psi_i(s - \sigma_{ij})) \right) \right| \right\} \\ &\le \sup_{t\in\mathbb{R}} \max_{i=1,n} \left\{ \int_{-\infty}^{t} e^{-a_i^-(t-s)} \left[\sum_{j=1}^{p} p_{ji}^+ \mu_j \left| \xi_j(s - \tau_{ji}) - \varphi_j(s - \tau_{ji}) \right| \right] \right\} \\ &+ \sup_{t\in\mathbb{R}} \max_{j=1,p} \left\{ \left| \int_{-\infty}^{t} e^{-b_j^-(t-s)} \left[\sum_{i=1}^{n} q_{ij}^+ \mu_i \left| \eta_i(s - \sigma_{ij}) - \psi_i(s - \sigma_{ij}) \right| \right] \right| \right\} \\ &\le \sup_{t\in\mathbb{R}} \max_{i=1,n} \left\{ \int_{-\infty}^{t} e^{-a_i^-(t-s)} \sum_{j=1}^{p} p_{ji}^+ \mu_j \|z_1 - z_2\| ds \right\} \\ &+ \sup_{t\in\mathbb{R}} \max_{j=1,p} \left\{ \int_{-\infty}^{t} e^{-b_j^-(t-s)} \sum_{i=1}^{n} q_{ij}^+ \mu_i \|z_1 - z_2\| ds \right\} \end{aligned}$$

$$= \left\{ \max_{i=\overline{1,n}} \left\{ \frac{\sum_{j=1}^{p} p_{ji}^{+}\mu_j}{a_i^{-}} \right\} + \max_{j=\overline{1,p}} \left\{ \frac{\sum_{i=1}^{n} q_{ij}^{+}\mu_i}{b_j^{-}} \right\} \right\} \|z_1 - z_2\| = r \|z_1 - z_2\|$$

Noting that $r < 1$, it is clear that T is a contraction mapping. Thus, T possesses a unique fixed point $z^* \in B^*$, that is $T(z^*) = z^*$. Then, by (2a) and (2b), $(x^*, y^*)^T$ satisfies (1a) and (1b). So, $(x^*, y^*)^T$ is a unique periodic solution of (1a) and (1b) in B^*. □

4 Global asymptotic stability of almost periodic solution

Set these:

$$u_i(t) = x_i(t) - x_i^*(t),$$
$$g_i(u_i(t-\sigma_{ij})) = f_i(u_i(t-\sigma_{ij}) + x_i^*(t)) - f_i(x_i^*(t)),$$
$$g_j(v_j(t-\tau_{ij})) = f_j\left(v_j(t-\tau_{ij}) + y_j^*(t)\right) - f_j\left(y_j^*(t)\right).$$

Then

$$|g_i(u_i(t-\sigma_{ij}))| \leq \mu_i |u_i(t-\sigma_{ij})|,$$
$$|g_j(v_j(t-\tau_{ij}))| \leq \mu_j |v_j(t-\tau_{ij})|,$$

for $i = 1, 2, ..., n$ and $j = 1, 2, ..., p$.

It is easy to see that system (1a) and (1b) can be reduced to the following system

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -a_i(t) u_i(t) + \sum_{j=1}^{p} p_{ji}(t) g_j(v_j(t-\tau_{ji})) \text{ (3a)}$$

$$\frac{\mathrm{d}v_j}{\mathrm{d}t} = -b_j(t) v_j(t) + \sum_{i=1}^{n} q_{ij}(t) g_i(u_i(t-\sigma_{ij})) \text{ (3b)}$$

Theorem 4.1. *[3].*

Assume that the signal functions f_i, $i = \overline{1, \max\{n,p\}}$ satisfy the hypotheses (H1) and (H2), suppose furthermore that (H3) and (H4) hold. If the system parameters satisfy the following conditions

(H5) $\sum_{j=1}^{p} \left(p_{ji}^{+} + \mu_i^2 q_{ij}^{+}\right) < 2a_i^{-}, \quad \sum_{i=1}^{n} \left(q_{ij}^{+} + \mu_i^2 p_{ji}^{+}\right) < 2b_i^{-},$

then the almost periodic solution of the system (1a), (1b) is global asymptotically stable.

Proof. Consider the Lyapunov functional $V(t)$ defined by

$$V(t) = V_1(t) + V_2(t),$$

$$V_1(t) = \sum_{i=1}^{n} u_i^2(t) + \sum_{j=1}^{p} v_j^2(t),$$

$$V_2(t) = \sum_{i=1}^{n} \sum_{j=1}^{p} \mu_j^2 p_{ji}^{+} \int_{t-\tau_{ji}}^{t} v_j^2(s)\, ds + \sum_{j=1}^{p} \sum_{i=1}^{n} \mu_i^2 q_{ij}^{+} \int_{t-\sigma_{ij}}^{t} u_i^2(s)\, ds.$$

By calculating the derivative of the $V_1(t)$ and $V_2(t)$ along the solutions of (3) and with (H5), we have that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$. Therefore, every solutions of (1a) and (1b) remains bounded for all $t \geq 0$, then the derivatives $\frac{\mathrm{d}u_i(t)}{\mathrm{d}t}$, $\frac{\mathrm{d}v_j(t)}{\mathrm{d}t}$ also remain bounded for all $t \geq 0$, which implies that $u_i(t)$, $\mathrm{v}_j(t)$ are uniformly continuous on $[0, +\infty)$.

It follows that $V(t)+\int_0^t \sum\limits_{i=1}^{n} u_i^2(s)\, r_i ds+\int_0^t \sum\limits_{j=1}^{p} v_j^2(s)\, s_j ds \le V(0)$. Thus $u_i(t),\ \mathrm{v}_j(t) \in L_1[0,+\infty)$.

In the end, we can get that $\lim\limits_{t\to\infty} u_i(t)=0,\ \lim\limits_{t\to\infty} v_j(t)=0.$ □

5 Example

Consider the following simple BAM networks with periodic coefficients and delays,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=-a_i(t)\,x_i(t)+\sum_{j=1}^{2} p_{ji}(t)\, f_j(y_j(t-2\pi))+I_i(t)$$

$$\frac{\mathrm{dy}_j}{\mathrm{d}t}=-b_j(t)\,y_j(t)+\sum_{i=1}^{2} q_{ij}(t)\, f_i(x_i(t-2\pi))+J_j(t)$$

where $I_i(t)=\sin t,\ J_j(t)=\cos t,\ \sigma_{ij}=\tau_{ji}=2\pi$, for $i,j\in\{1,2\}$.
To take $f_i(x)=\frac{1}{2}(|x+1|-|x-1|)$,$\mu_i=1,\ i\in\{1,2,3,4\}$.
Again, take

$$(a_1(t),a_2(t))^T=(3+2\cos t,3+2\sin t)^T$$
$$(b_1(t),b_2(t))^T=(3+2\sin t,3+2\cos t)^T$$

we obtain

$$\left(a_1^-,a_2^-\right)^T=(1,1,1)^T,\quad \left(b_1^-,b_2^-\right)^T=(1,1,1)^T$$
$$\left(a_1^+,a_2^+\right)^T=(5,5,5)^T,\quad \left(b_1^+,b_2^+\right)^T=(5,5,5)^T$$
$$M[a_i]>0,\quad M[b_j]>0.$$

Let

$$\begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t)\end{pmatrix}=\begin{pmatrix} 0.05\cos t & 0.05\cos t \\ 0.1\sin t & 0.15\sin t\end{pmatrix}$$

$$\begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t)\end{pmatrix}=\begin{pmatrix} 0.35\cos t & 0.1\cos t \\ 0.15\sin t & 0.15\sin t\end{pmatrix}$$

Then

$$\begin{pmatrix} q_{11}^+(t) & q_{12}^+(t) \\ q_{21}^+(t) & q_{22}^+(t)\end{pmatrix}=\begin{pmatrix} 0.05 & 0.05 \\ 0.1 & 0.15\end{pmatrix}$$

$$\begin{pmatrix} p_{11}^+(t) & p_{12}^+(t) \\ p_{21}^+(t) & p_{22}^+(t)\end{pmatrix}=\begin{pmatrix} 0.35 & 0.1 \\ 0.15 & 0.15\end{pmatrix}$$

Moreover, $r=\max\limits_{i=1,2}\left\{\frac{\sum\limits_{j=1}^{2} p_{ji}^+\mu_j}{a_i^-}\right\}+\max\limits_{j=1,2}\left\{\frac{\sum\limits_{i=1}^{2} q_{ij}^+\mu_i}{b_j^-}\right\}=0.65<1.$

Set $\lambda_i=1,\quad i=1,2,3,4,$, we get

$$\lambda_1\left(-2a_1^-+\sum_{j=1}^{2} p_{j1}^+\mu_j\right)+\sum_{j=1}^{2}\lambda_{2+j}q_{1j}^+\mu_1=-1.4<0$$

$$\lambda_2\left(-2a_2^-+\sum_{j=1}^{2} p_{j2}^+\mu_j\right)+\sum_{j=1}^{2}\lambda_{2+j}q_{2j}^+\mu_2=-1.5<0$$

$$\lambda_3\left(-2b_1^-+\sum_{i=1}^{2} q_{i1}^+\mu_i\right)+\sum_{i=1}^{2}\lambda_i p_{1i}^+\mu_1=-1.4<0$$

$$\lambda_4\left(-2b_2^- + \sum_{i=1}^{2} q_{i2}^+\mu_i\right) + \sum_{i=1}^{2}\lambda_i p_{2i}^+\mu_2 = -1.5 < 0.$$

Thus, it follows from Theorem 3.1 and Theorem 4.1 that the unique periodic solution is globally exponential stable.

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