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ABOUT HERON'S FORMULA FOR CUBE ROOT

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Abstract: *In this paper we extend the Heron's method for cube root to nth root and give an automatic error control using interval arithmetic.*

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1 Introduction

We cite from [1]:" also in Book III of *Metrica*, Heron gives a method to find the cube root of a number. In particular Heron finds the cube root of 100 and the authors of [3] give a general formula for the cube root of N which Heron seems to have used in his calculation:

$a + bd/(bd + aD)(b - a)$,

where $a^3 < N < b^3, d = N - a^3, D = b^3 - N$.

In [3] it is remarked that this is a very accurate formula, but, unless a Byzantine copyist is to be blamed for an error, they conclude that Heron might have borrowed this accurate formula without understanding how to use it in general."

In [3] the authors prove that for $b = N/a^2$ Heron formula has the order of convergence 3.

In the following we show that the Heron formula can be obtained from the secant method. We find also a formula that is obtained using a Newton - Raphson method and with the order of convergence 3.We generalize the Heron formula for n-th root of a positive number.

2 Main results

In the following we note $N = \alpha^3$ a positive number. Let $x < \alpha$ a approximation of the cube root of N, and $f : (0, \infty) \to \mathbb{R}$ a real function defined by:

$$f(x) = \frac{x^3 - \alpha^3}{x}. \tag{1}$$

Theorem 2.1. *The Heron formula is the secant method formula for the function f and the points a and b.*

Proof. The secant method formula is:

$$y = a - \frac{f(a)}{f(b) - f(a)}(b - a)$$

where y is the new approximation for the root of f. If we replace 1 in the above formula it results the equality:

$$y = a + \frac{b(N - a^3)}{a(b^3 - N) + b(N - a^3)}(b - a).$$

The right side of this equality is the Heron formula. □

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Remark 2.2. *If we simplify the Heron formula it results:*

$$y = \frac{a^2b^2 + Na + Nb}{a^2b + ab^2 + N} = \frac{a^2b^2 + \alpha^3 a + \alpha^3 b}{a^2b + ab^2 + \alpha^3}$$

and:

$$y - \alpha = \frac{a^2b^2 + \alpha^3 a + \alpha^3 b}{a^2b + ab^2 + \alpha^3} - \alpha = (b - \alpha)(a - \alpha)\frac{-\alpha^2 + ab}{\alpha^3 + ab^2 + a^2b} \tag{2}$$

If $\alpha - a = b - \alpha = \varepsilon$ from the above equalities results:

$$y - \alpha = \frac{\varepsilon^4}{3\alpha^3 - 2\alpha\varepsilon^2} \tag{3}$$

e.g. in this case the order of convergence is 4. *In [3] from 2 results that the order of convergence is* 3.

Example 2.3. *For $\alpha = \sqrt[3]{2}$ and $\varepsilon = 1/10$ we obtain from 3 (using Mathematica) $y = 1.2599377868523566469$ and $y - \alpha = 0.000016736957483482115471$.*

In the following we use the function f defined by 1. We consider the sequence $(x_k)_{k=0,1,\ldots}$ defined by the Newton-Raphson method:

$$x_0 = a < \alpha, |a - \alpha| < 1 \tag{4}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{5}$$

Theorem 2.4. *The sequence defined by 4,5 converge to $\alpha = \sqrt[3]{N}$ and the order of convergence is* 3.

Proof. The convergence results from convergence of the Newton-Raphson method. If we calculate the right side of 5 it result:

$$x_{k+1} = \frac{x_k(x_k^3 + 2\alpha^3)}{2x_k^3 + \alpha^3} \tag{6}$$

Let now $g : (0, \infty) \to \mathbb{R}$, defined by:

$$g(x) = \frac{x(x^3 + 2\alpha^3)}{2x^3 + \alpha^3} - \alpha$$

We have $g(\alpha) = 0,\ g'(\alpha) = 0,\ g''(\alpha) = 0, g'''(\alpha) = \frac{1}{4\alpha^2}$, so:

$$g(x) = \frac{4}{3!\alpha^2}(x - \alpha)^3 + \frac{(x - a)^4}{4!}g^{(iv)}(\xi)$$

where ξ is a point between x and α. From the above formula and 6 it follows:

$$x_{k+1} - \alpha = \frac{4}{3!\alpha^2}(x_k - \alpha)^3 + \frac{(x_k - a)^4}{4!}g^{(iv)}(\xi_k)$$

From this formula it results that :

$$\lim_{k\to\infty}\left|\frac{x_{k+1} - \alpha}{(x_k - \alpha)^3}\right| = \frac{4}{3!\alpha^2}$$

e.g. the order of convergence is 3. □

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Remark 2.5. *It is easy to calculate that*

$$g(x) = \frac{(x-\alpha)^3 (x+\alpha)}{2x^3+\alpha^3}.$$

Example 2.6. *Let* $\alpha = \sqrt[3]{2}$, $x_0 = 1.2$ *. We obtain* $x_3 = 1.2599210498948731648$ *and* $x_3^3 = 2$ *with twenty decimal digits.*

Now we generalize the algorithm to find the nth root of a real positive number. In the following we note $N = \alpha^n$ and $f_n : (0, \infty) \to \mathbb{R}$ the real function defined by:

$$f_n(x) = \frac{x^n - \alpha^n}{x^{\frac{n-1}{2}}}$$

and consider the sequence $(x_k)_{k=0,1,\dots}$ defined by the Newton-Raphson method:

$$x_0 = a < \alpha, |a - \alpha| < 1 \tag{7}$$

$$x_{k+1} = x_k - \frac{f_n(x_k)}{f_n'(x_k)}. \tag{8}$$

The generalization of the theorem 2.4 is:

Theorem 2.7. *The sequence defined by 7,8 converge to* $\alpha = \sqrt[n]{N}$ *and the order of convergence is* 3.

Proof. The convergence results from convergence of the Newton-Raphson method. If we calculate the right side of 8 we obtain:

$$x_{k+1} = x_k \frac{(n-1)x_k^n + (n+1)\alpha^n}{(n+1)x_k^n + (n-1)\alpha^n} \tag{9}$$

Let now $g_n : (0, \infty) \to \mathbb{R}$, defined by:

$$g_n(x) = x\frac{(n-1)x^n + (n+1)\alpha^n}{(n+1)x^n + (n-1)\alpha^n} - \alpha.$$

We have $g_n(\alpha) = 0$, $g_n'(\alpha) = 0$, $g_n''(\alpha) = 0$, $g_n'''(\alpha) = \frac{n^2-1}{2\alpha^2}$, so:

$$g_n(x) = \frac{n^2-1}{3!\alpha^2}(x-\alpha)^3 + \frac{(x-a)^4}{4!} g_n^{(iv)}(\xi)$$

where ξ is a point between x and α. From the above formula and 9 it follows:

$$x_{k+1} - \alpha = \frac{n^2-1}{3!\alpha^2}(x_k-\alpha)^3 + \frac{(x_k-a)^4}{4!} g_n^{(iv)}(\xi_k)$$

It follows that:

$$\lim_{k\to\infty} \left| \frac{x_{k+1}-\alpha}{(x_k-\alpha)^3} \right| = \frac{n^2-1}{3!\alpha^2},$$

e.g. the order of convergence is 3. □

Example 2.8. *For* $N = 3$, $n = 4$, $x_0 = 1.2$ *we obtain* $x_3 = 1.3160740129524924608$ *which have the twenty decimals of* $\sqrt[4]{3}$.

Remark 2.9. *The formula 9 for* $n = 2$ *can be founded in [2].*

Now we apply to the above method interval analysis, to find an automatic error bound. For an closed and bounded interval X of real numbers we note with $m(X)$ the middle of the interval and with $w(X)$ the length. We also noted with F_n' the natural interval extension of function f' $\left(F_n'(X) = \left(\frac{n+1}{2}X^{\frac{n-1}{2}} + N\frac{\frac{n-1}{2}}{X^{\frac{n+1}{2}}}\right)\right)$.Following an idea from [4] we consider for the interval X the interval:

$$K(X) = m(X) + \left(-\frac{1}{F_n'(X)}\right) f_n(m(X)).$$

and for $X_0 = [a, b]$, with $0 < a < \sqrt[n]{N} < b$ we define the sequence:

$$X_{k+1} = X_k \cap N(X_k), k \geq 0.$$

Theorem 2.10. *The sequence* $(X_k)_{k\in\mathbb{N}}$ *has the properties:*
for each k $\sqrt[n]{N} \in X_k$, $\lim_{k\to\infty} w(X_k) = 0$, $w(X_{k+1}) \leq c(w(X_k))^2$.

Proof. Because $0 \notin F_n'(X_0)$ we can apply Theorem 1, pag. 72 from [4] and the theorem results. □

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