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GENERALIZATION OF PÁL ERDÖS INEQUALITY

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Let's consider any triangle ABC with the lengths a,b,c and the point D in his interior. We point out with M,N and P the perpendicular feet coming from D to the sides BC, CA, AB and with *M1, N1* i *P1* the feet of the bisectoars of the angles $\angle BDC$, $\angle CDA$, and $\angle ADB$on the sides BC, CA i AB. We also note:

$$\begin{array}{lll} AD = x & DM = u & DM_1 = u_1 \\ BD = y & DN = v & DN_1 = v_1 \\ CD = z & DP = w & DP_1 = w_1 \end{array}$$

In 1935, In the *American Mathematical Monthly* (AMM) Pál Erdös (26.03.1913 – 20.09.1996) published, as an open problem, the following inequality:

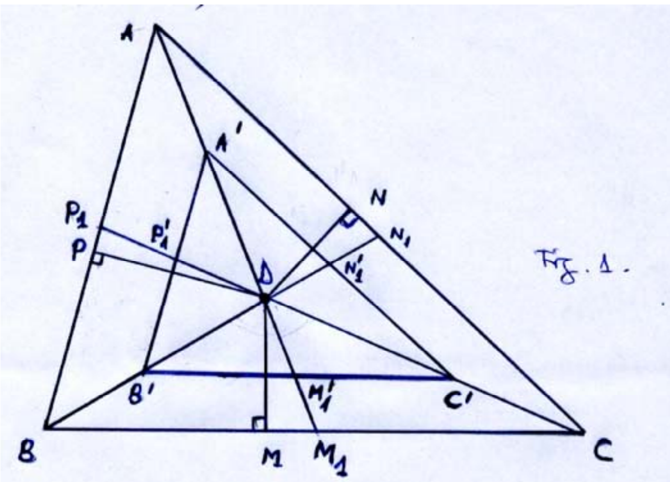
$$x + y + z \geq 2(u + v + w) \tag{1}$$

L. J. Mordell published in 1937, also in AMM, the solution of this problem, which is known in mathematical literature like Erdös – Mordell theoreme (problem). In 1945, D. K. Kazarinoff published a new proof of this problem based on the reflexion principle.

The American mathematician D. Barrow published the following inequality:

$$x + y + z \geq 2(u_1 + v_1 + w_1) \tag{2}$$

stronger inequality than (1), taking into account $u_1 \geq u$, $v_1 \geq v$, $w_1 \geq w$ (see figure 1)



To prove the relation (2) we consider $Q = x + y + z - 2(u_1 + v_1 + w_1)$ and show that we have $Q \geq 0$. Noting $\alpha = m(\angle BDC)$, $\beta = m(\angle CDA)$ i $\chi = m(\angle ADB)$ we'll have:

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$$u_1 = \frac{2yz}{y+z}\cos\frac{\alpha}{2} \tag{3}$$

And taking into account the means inequality follows:

$$u_1 \leq \sqrt{yz}\cos\frac{\alpha}{2} \tag{4}$$

Analogously we obtain

$$\begin{cases} v_1 \leq \sqrt{zx}\ \cos\frac{\beta}{2} \\ w_1 \leq \sqrt{xy}\ \cos\frac{\chi}{2} \end{cases} \tag{5}$$

and using results in $\frac{\chi}{2} = \pi - \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)$ and

$$\cos\frac{\chi}{2} = \sin\frac{\alpha}{2}\sin\frac{\beta}{2} - \cos\frac{\alpha}{2}\cos\frac{\beta}{2}.$$

On these conditions,the expression Q becomes successively:

$$Q \geq x+y+z-2\sqrt{yz}\cos\frac{\alpha}{2} - 2\sqrt{zx}\cos\frac{\beta}{2} - 2\sqrt{xy}\cos\frac{\chi}{2} =$$

$$= x+y+z-2\sqrt{yz}\cos\frac{\alpha}{2} - 2\sqrt{zx}\cos\frac{\beta}{2} + 2\sqrt{xy}\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - 2\sqrt{xy}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} =$$

$$= \left(\sqrt{x}\cos\frac{\beta}{2} + \sqrt{y}\cos\frac{\alpha}{2} - \sqrt{z}\right)^2 + \left(\sqrt{x}\sin\frac{\beta}{2} - \sqrt{y}\sin\frac{\alpha}{2}\right)^2 \geq 0$$

We have equality for: $\begin{cases} x+y = 2\sqrt{xy} \\ y+z = 2\sqrt{yz} \\ x+z = 2\sqrt{xz} \\ \sqrt{x}\cos\frac{\beta}{2} + \sqrt{y}\cos\frac{\alpha}{2} - \sqrt{z} = 0 \\ \sqrt{x}\sin\frac{\beta}{2} - \sqrt{y}\sin\frac{\alpha}{2} = 0 \end{cases}$, equalities that show that the triangle ABC is equilateral and the point D is the circumcentre.

We have the following:

Theorem: If *m*,*n* and *p are positive real numbers*, we have:

$$mx + ny + pz \geq 2\left[\frac{np(y+z)}{ny+pz}u_1 + \frac{pm(z+x)}{pz+mx}v_1 + \frac{mn(x+y)}{mx+ny}w_1\right] \tag{6}$$

Proof. On the vectors radius *DA, DB, DC* we choose the points *A', B', C'* such that $DA' = mx$, $DB' = ny$, $DC' = pz$. The point D is in the triangle interior *A'B'C'* and noting M_1', N_1', P_1' the intersections of *DM1* with *B'C'*, *DN1* with *C'A'* and *DP1* with *A'B'*. Calculating the length of the bisector DM_1' and taking into account (3) on obtaining:

$$DM_1' = \frac{2DB' \cdot DC'}{DB' + DC'}\cos\frac{\alpha}{2} = \frac{2ny \cdot pz}{ny+pz}\cos\frac{\alpha}{2} = \frac{np(y+z)}{ny+pz}u_1 \tag{7}$$

Analoguously we have:

$$DN_1' = \frac{pm(z+x)}{pz+mx}v_1$$

$$DP_1' = \frac{mn(x+y)}{mx+ny}v_1 \tag{8}$$

Applying the inequality (2) for the triangle *A'B'C'* we have:

$$DA^{'} + DB^{'} + DC^{'} \geq 2(DM_1^{'} + DN_1^{'} + DP_1^{'})$$

and regarding the relations (7) and (8) we obtain (6) with equality if and only if when we have $mx = ny = pz$ and the point D is the *Fermat point* in triangle ABC (meaning $m(\angle ADB) = m(\angle BDC) = m(\angle CDA) = 120°$).

Because of that $u_1 \geq u,\ v_1 \geq v,\ w_1 \geq w$from (6) follows:

$$mx + ny + pz \geq 2\left[\frac{np(y+z)}{ny+pz}u + \frac{pm(z+x)}{pz+mx}v + \frac{mn(x+y)}{mx+ny}w\right] \tag{9}$$

□

Inequalities (6) and (9) represent generalizations of D. Barrow and P. Erdös inequalities.

Applications

1. For $mx = ny = pz$using the mean inequality we obtain the following:

$$xyz \geq (u+v)(v+w)(w+u) \tag{10}$$

$$xyz \geq (u_1+v_1)(v_1+w_1)(w_1+u_1) \tag{11}$$

Remark: Inequality (10) belongs to A. Oppenheim and was published in AMM in 1960.

From (10) and (11) immediately follows:

$$xyz \geq 8uvw \tag{12}$$

$$xyz \geq 8u_1v_1w_1 \tag{13}$$

2. For $m = \frac{yz}{x}$, $n = \frac{zx}{y}$, $p = \frac{xy}{z}$ we have:

$$xy + yz + zx \geq 2(xu + yv + zw) \tag{14}$$

$$xy + yz + zx \geq 2(xu_1 + yv_1 + zw_1) \tag{15}$$

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