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# FUNCTIONS WITH RESTRICTED CAUCHY KERNEL IMAGE

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***Abstract:*** *In this paper we give a characterization of the functions whose Cauchy kernel image belongs to a subgroup or a linear subspace. The results are in connection with Hyers-Ulam stability of functional equations.*

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## 1 Introduction

Let $(G,+)$ and $(H,+)$ be two commutative groups and $f : G \to U$ a function.

The function $C_f : G \times G \to H$,

$$C_f(x,y) = f(x+y) - f(x) - f(y), \quad (x,y) \in G \times G,$$

is called the Cauchy difference [4] or Cauchy kernel [5] of function $f$.

If $H_0 \subset H$ is an arbitrary subset of $H$, a condition of the form $C_f(x,y) \in H_0$, $(x,y) \in G \times G$, restricts the image of the Cauchy kernel from the set $H$ to the set $H_0$. If $H$ is a linear normed space and $B_\varepsilon$ is the ball of center zero and radius $\varepsilon$, i.e. $B_\varepsilon = \{y \in H \mid \|y\| < \varepsilon\}$ then the condition $C_f(x,y) \in B_\varepsilon$, $(x,y) \in G \times G$, leads to the stability problem for the Cauchy functional equation in Hyers-Ulam sense [3], [6]. The first result on the stability of functional equation given by Hyers in 1941, is the next one [3]:

**Theorem 1.1.** *Let $X$ be a linear normed space, $Y$ a Banach space and $\varepsilon > 0$. Then for every function $f : X \to Y$ satisfying*

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon, \quad x, y \in X,$$

*there exists a unique additive function $g : X \to Y$ such that*

$$\|f(x) - g(x)\| < \varepsilon, \quad x \in X.$$

In this paper we will characterize the functions whose Cauchy kernel image is restricted to a subgroup or a linear space. Contributions on this topic are given in [1], [2], [4].

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## 2 Functions with Cauchy kernel image in a subgroup

Let $(G,+)$ and $(H,+)$ be two commutative groups, $(H_0,+)$ be a subgroup of $(H,+)$, $(H/H_0,+)$ the quotien group and $p : H \to H/H_0$, $p(x) = x + H_0$, $x \in H$, the canonical projection.

**Theorem 2.1.** *The function $f : G \to H$ satisfies the condition:*

$$f(x+y) - f(x) - f(y) \in H_0, \quad (x,y) \in G \times G$$

*if and only if the function $g : G \to H/H_0$, $g = p \circ f$ is a group homomorphism.*

*Proof.* The canonical projection $p : H \to H/H_0$ is a surjective homomorphism of group with $\ker p = H_0$.

The condition

$$f(x+y) - f(x) - f(y) \in H_0 = \ker p$$

is equivalent to

$$p(f(x+y) - f(x) - f(y)) = \widehat{0},$$

where $\widehat{0} = 0 + H_0 = H_0$ is the zero element of the quotient group $H/H_0$. Since $p$ is a homomorphism of group we get:

$$p(f(x+y)) - p(f(x)) - p(f(y)) = \widehat{0}, \quad (x,y) \in G \times G$$

or

$$(p \circ f)(x+y) = (p \circ f)(x) + (p \circ f)(y), \quad (x,y) \in G \times G.$$

By the last relation follows that $p \circ f$ is a homomorphism of groups. □

**Corolar 2.2.** *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition*

$$f(x+y) - f(x) - f(y) \in \mathbb{Z}, \text{ for every } (x,y) \in \mathbb{R} \times \mathbb{R}$$

*if and only if the function $h : \mathbb{R} \to \mathbb{C}^*$,*

$$h(x) = e^{2\pi i f(x)}, \quad x \in \mathbb{R},$$

*is a homomorphism of groups.*

*Proof.* The condition $h(x+y) = h(x)h(y)$ leads to

$$e^{2\pi i(f(x+y)-f(x)-f(y))} = 1 \quad \text{or} \quad f(x+y) - f(x) - f(y) \in \mathbb{Z}.$$

□

**Remark 2.3.** *The group $(\mathbb{R}/\mathbb{Z},+)$ is isomorphic with the group $(U,\cdot)$, where*

$$U = \{z \in \mathbb{C} \mid |z| = 1\}.$$

*The function $\varphi : \mathbb{R} \to \mathbb{C}^*$, $\varphi(x) = e^{2\pi i x}$, $x \in \mathbb{R}$ is a homomorphism from the group $(\mathbb{R},+)$ to the group $(\mathbb{C}^*,\cdot)$ with $\ker \varphi = \mathbb{Z}$. In view of the isomorphism theorem the group $\mathbb{R}/\ker \varphi$ and $U = \mathrm{Im}\varphi$ are isomorphic. If $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the canonical projection $p(x) = x + \mathbb{Z}$, $x \in \mathbb{R}$, the function $g = p \circ f$, $g : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a homomorphism of groups if and only if the function $h : \mathbb{R} \to \mathbb{C}^*$, $h = \varphi \circ g$ is a homomorphism of groups.*

## 3 Functions with Cauchy kernel image in a linear subspace

Let $(V,+)$ and $(W,+)$ be two linear spaces over the same field and $W_0$ a linear subspace of $W$.

**Theorem 3.1.** *The function $f : V \to W$ satisfies the condition:*

$$f(x+y) - f(x) - f(y) \in W_0, \quad (x,y) \in V \times V,$$

*if and only if there exists a linear function $T : W \to W$ with the property* $\ker T = W_0$, *therefore the function $g : V \to W$, $g = T \circ f$ be additive.*

*Proof.* We choose a basis $\{e_i \mid i \in I\}$ in $W_0$ which will be extended to a basis $\{e_i \mid i \in I\} \cup \{e'_j \mid j \in J\}$ in $W$. If we denote by $W_1 = Span\{e'_j \mid j \in J\}$ then $W$ is the direct sum of $W_0$ and $W_1$, i.e. $W = W_0 \oplus W_1$.

The function $T : W \to W$ defined by $T(w_0 + w_1) = w_1$, $w_0 \in W_0$, $w_1 \in W_1$ is a linear function with $\ker T = W_0$. The condition

$$f(x+y) - f(x) - f(y) \in W_0 = \ker T$$

is equivalent to

$$T(f(x+y) - f(x) - f(y)) = 0$$

or

$$(T \circ f)(x+y) = (T \circ f)(x) + (T \circ f)(y).$$

This show that $g = T \circ f$ is linear function. □

**Corolar 3.2.** *The function $f : \mathbb{C} \to \mathbb{C}$ satisfies the condition*

$$f(z_1 + z_2) - f(z_1) - f(z_2) \in \mathbb{R}, \quad z_1, z_2 \in \mathbb{C},$$

*if and only if there exists a function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and two additive functions $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ such that:*

$$f(x+iy) = u(x,y) + i(a(x) + b(y)), \quad x, y \in \mathbb{R}.$$

*Proof.* The group $(\mathbb{C},+)$ is a linear space over $\mathbb{R}$ and $T : \mathbb{C} \to \mathbb{C}$, $T(x+iy) = y$, $x, y \in \mathbb{R}$, is a linear function with $\ker T = \mathbb{R}$. The function $f$ satisfies the condition of Theorem 3.1, for $W_0 = \mathbb{R}$ if and only if the function $g = T \circ f : \mathbb{C} \to \mathbb{C}$ is additive. Taking account of Theorem 1, from [5], the function $g : \mathbb{C} \to \mathbb{C}$ is additive if and only if there exist additive functions $f_{11}, f_{12}, f_{21}, f_{22} : \mathbb{R} \to \mathbb{R}$ such that:

$$g(x+iy) = f_{11}(x) + f_{12}(y) + i(f_{21}(x) + f_{22}(y)), \quad x, y \in \mathbb{R}.$$

If $f(x+iy) = f_1(x+iy) + if_2(x+iy)$, $x, y \in \mathbb{R}$ where $f_1, f_2 : \mathbb{C} \to \mathbb{R}$, then

$$(T \circ f)(x+iy) = f_2(x+iy), \quad x, y \in \mathbb{R}.$$

From $g = T \circ f$ follows:

$$f_{11}(x) + f_{12}(y) + i(f_{21}(x) + f_{22}(y)) = f_2(x+iy) \in \mathbb{R}, \quad x, y \in \mathbb{R}.$$

Hence $f_{21} = f_{22} = 0$ and $f_2(x+iy) = f_{11}(x) + f_{12}(y)$, $x, y \in \mathbb{R}$ and then

$$f(x+iy) = f_1(x+iy) + i(f_{11}(x) + f_{12}(y))$$

$$= u(x,y) + i(a(x) + b(y)), \quad x, y \in \mathbb{R},$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an arbitrary function and $a, b : \mathbb{R} \to \mathbb{R}$ are additive functions. □

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