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COMPACTNESS OF SOME CLASSES OF CONVEX FUNCTIONS

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Abstract: *After two criteria of equicontinuity, we shall study the compactness and some consequences on the uniform convergence in the class of the solutions of the Dirichlet problem for the Monge-Ampère equation.*

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1 Criteria of equicontinuity

In this work $(E, \|\cdot\|)$ is a real normed space. If A is a nonvoid subset of E then $C(A) := \{u \in \mathbb{R}^A : u \text{ is continuous}\}$, and if A is convex then $\mathcal{U}(A) := \{u \in \mathbb{R}^A : u \text{ is convex}\}$.

D denotes an open nonvoid subset of a given topological space. If $A \subset X$ then, $C(A; L)$ denotes the continuous functions on A with values in L. From now on for each sets A, B and $H \subset B^A$, and for each $x \in A$ we have $H(x) := \{h(x) : h \in H\}$.

Proposition 1.1. *Assume that X and L are Hausdorff uniform spaces, Y is a dense set in X, and $H \subset C(X; L)$ such that H is uniformly equicontinuous on Y. We have the following assertions:*

(i). H is uniformly equicontinuous on X.

(ii). If for each $y \in Y$, $H(y)$ is a precompact (respectively a relatively compact) subset of L, then H is a precompact (respectively a relatively compact) subset of $C(X; L)$ with respect to the topology of the precompact convergence.

Remark 1.2. (i). In the framework of Proposition 1, if we suppose that X is precompact, then H is precompact (respectively a relatively compact) subset of $C(X; L)$ with respect to the topology of the uniform convergence on X.

(ii). We shall apply the previous proposition in the case of on open subset D of X and $H \subset C(\overline{D}; L)$ uniformly equicontinuous on D.

Proposition 1.3. *Suppose that X is a Hausdorff topological space, Y is a nonvoid subset of X, L is a topological vector lattice and $H \subset C(X; L)$ such that*

$$\exists u, v \in C(X; L) : u|_{X \setminus Y} = 0_L \text{ and } \forall h \in H, |h - v| \leq |u|. \tag{1.1}$$

We have the following assertions:

(i). H is equicontinuous on $X \setminus Y$.

(ii). If H is equicontinuous on Y and for each $y \in Y$, $H(y)$ is a precompact (respectively a relatively compact) subset of L, then H is a precompact (respectively a relatively compact) subset of $C(X; L)$ with respect to the topology of the compact convergence on X.

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Remark 1.4. (i). Assume that in the conditions of the previous proposition the family H has the following property:

$$\exists u \in C(X;L) : u|_{X\setminus Y} = 0_L \text{ and } \forall h_1, h_2 \in H, |h_1 - h_2| \leq |u|. \tag{1.2}$$

It follows that H has also the properties stated in the conclusion of the previous proposition.

(ii). The hypothesis of the second point of Proposition 3 is satisfied when H is a precompact (respectively a relatively compact) subset of $C(Y, L)$ with respect to the topology of the compact convergence on Y.

(iii). We shall apply the result of Proposition 3 in the following case: $D \subset X$ is a nonvoid open subset, $H \subset C(\overline{D}; L)$ and either:

$$\exists u, v \in C(\overline{D}, L) : u|_{\partial D} = 0_L \text{ and } \forall h \in H, |h - v| \leq |u|. \tag{1.3}$$

or

$$\exists u \in (\overline{D}; L) : u|_{\partial D} = 0_L \text{ and } \forall h_1, h_2 \in H, |h_1 - h_2| \leq |u|. \tag{1.4}$$

Corollary 1.5. *Given X a compact space and u, v, H as in Proposition 3 (respectively u, H as in Remark 4 (i).) it follows:*

(i). H is equicontinuous on $X \setminus Y$.

(ii). If H is equicontinuous on Y and $L = \mathbb{R}$, H is relatively compact with respect to the topology of the uniform convergence on X.

2 Some backgrounds of the convex functions

This section is devoted to defining and recalling the basic notions and results (from [1], [2], [6] or [13]) which are used in this work.

Lemma 2.1. *Suppose D is convex, $H \subset \mathcal{U}(D)$ and $a \in D$ such that $H(a) := \{h(a) : h \in H\}$ is bounded. The following assertions are equivalent:*

(i). For each $x \in D$, $H(x)$ is upper bounded,

(ii). For each $x \in D$, $H(x)$ is bounded.

Theorem 2.2. *([1] and [2]) Given D an open convex subset of E and $H \subset \mathcal{U}(D)$ such that for each $x \in D$, $H(x)$ is bounded then for every $a \in D$ the following assertions are equivalent:*

(i). H is equicontinuous at a.

(ii). H is upper bounded on a neighbourhood of a.

Corollary 2.3. *For each family H as in the previous proposition the following assertions are equivalent:*

(i). H is equicontinuous on D.

(ii). H is upper bounded on an open nonvoid subset of D.

Proposition 2.4. *Take $H \subset \mathcal{U}(D)$ where D (is open convex subset of E), V an open nonvoid subset of D and
$r \in (0, \infty)$ such that $V + \overline{B}(0_E, r) \subset D$ and H is bounded on $V + \overline{B}(0_E, r)$. Then H is equi-Lipschitz family on V (particularly H is uniformly equicontinuous on V).*

Corollary 2.5. *Being given D a nonvoid open convex subset of $\mathbb{R}^k$ and $H \subset \mathcal{U}(D)$ such that for each $x \in D$, $H(x)$ is bounded we have the following assertions*

(i). H is equicontinuous on D.

(ii). For each K a compact subset of D: (a). H is bounded on K. (b). H is equi-Lipschitz family on K (particularly H is uniformly equicontinuous on K).

Proposition 2.6. *If $H \subset \mathcal{U}(\overline{D}) \cap C(\overline{D})$ (D open convex subset of E) satisfies the condition (1.1) then we have the following assertions:*

(i). H is equicontinuous on $\overline{D}$.

(ii). H is relatively compact in $C(\overline{D})$ with respect to the topology of the compact convergence.

(iii). Suppose D relatively compact. It follows that H is relatively compact in $C(\overline{D})$ with respect to the topology of the uniform convergence.

From now on in this section $E = \mathbb{R}^k$ ($k \in \mathbb{N}^\star$), D is strictly convex and λ is the Lebesgue measure on $\mathbb{R}^k$. Moreover, $\mathcal{M}_+(D)$ (respectively $b\mathcal{M}_+(D)$) is the set of positive (respectively positive and bounded) Radon measures on D, and $L^\infty(D)$ denotes the space of λ-essentially bounded functions on D.

Theorem 2.7. ([6]) *(i). For each $u \in \mathcal{U}(D)$,there exists $\nu_u \in \mathcal{M}_+(D)$ such that if u is twice continuously differentiable and $K \subset D$ is compact*

$$\nu_u(K) = \int\limits_K \det(D^2u)d\lambda.$$

(ii). For each $u, v \in \mathcal{U}(D)$ and $\alpha \in \mathbb{R}_+^\star$ we have:
(a) $\nu_{u+v} \geq \nu_u + \nu_v$, (b)$\nu_{\alpha \cdot u} = \alpha^k \cdot \nu_u$.

(iii). If $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(D)$ and $(u_n)_{n\in\mathbb{N}^\star} \to u_0$ uniformly on the compact subsets of D, then $(\nu_{u_n})_{n\in\mathbb{N}^\star} \to (\nu_{u_0})$ vaguely on U.

Remark 2.8. (i). For each $u \in \mathcal{U}(D)$, ν_u is called the curvature measure of u.

(ii). If $\mu \in b\mathcal{M}_+(D)$, $\varphi \in C(\partial D)$ and $u \in \mathcal{U}(\overline{D})$ are defined as it follows:

$$\nu_u = \mu \text{ and } u|_{\partial D} = \varphi,$$

then u is called the solution of the Dirichlet problem for the Monge-Ampère equation (i.e., the solution of the Dirichlet problem).

Proposition 2.9. ([6]) *For each $u, v \in \mathcal{U}(D)$, we have the following assertions:*

(i). (The minimum principle). If $\nu_u \leq \nu_v$ and $(\mathrm{sci}_D u)|_{\partial D} \geq (\mathrm{sci}_D v)|_{\partial D}$, then $u \geq v$.
(Here $\mathrm{sci}_D u : \overline{D} \to \overline{\mathbb{R}}$, $\forall a \in \overline{D}, (\mathrm{sci}_D u)(a) := \liminf\limits_{D \ni x \to a} u(x)$).

(ii). (The boundedness principle). Suppose $m \in \mathbb{R}$ is such that $m \leq (\mathrm{sci}_D u)|_{\partial D}$. Then

$$u \geq m - (\mathrm{diam} D)\sqrt[k]{\frac{\nu_u(D)}{\omega_k}}, \text{ where } \omega_k := \lambda(B(0_k, 1).$$

Theorem 2.10. *(the existence of the solution of the Dirichlet problem)([6] and [11]). For each $\mu \in b\mathcal{M}_+(D)$ and $\varphi \in C(\partial D)$ there is one and only one function $M(\mu;\varphi) \in \mathcal{U}(\overline{D}) \cap C(\overline{D})$ such that:*

$$\nu_{M(\mu;\varphi)} = \mu \text{ and } (M(\mu;\varphi))|_{\partial D} = \varphi.$$

Proposition 2.11. *([6]). Take $\mu_1, \mu_2 \in b\mathcal{M}_+(D)$ and $\varphi_1, \varphi_2 \in C(\partial D)$. Then:*

(i). $M(\mu_1 + \mu_2; \varphi_1 + \varphi_2) \geq M(\mu_1;\varphi_1) + M(\mu_2;\varphi_2)$.

(ii). If $\mu_1 \leq \mu_2$ and $\varphi_1 \geq \varphi_2$, it follows that $M(\mu_1;\varphi_1) \geq M(\mu_2;\varphi_2)$.

(iii). $\inf \varphi_1 - (\mathrm{diam} D)\sqrt[k]{\frac{\mu_1(D)}{\omega_k}} \leq M(\mu_1, \varphi_1) \leq \sup \varphi_1$.

Corollary 2.12. *For each* $\mu, \mu_1, \mu_2 \in b\mathcal{M}_+(D)$ *and* $\varphi, \varphi_1, \varphi_2 \in C(\partial D)$ *we have the following sentences:*

(i). $|M(\mu_1;\varphi_1) - M(\mu_2;\varphi_2)| \leq -M(|\mu_1-\mu_2|; -|\varphi_1-\varphi_2|)$.

(ii). $|M(\mu_1;\varphi) - M(\mu_2;\varphi)| \leq -M(|\mu_1-\mu_2|; 0)$.

(iii). $|M(\mu;\varphi_1) - M(\mu;\varphi_2)| \leq -M(0; -|\varphi_1-\varphi_2|)$.

3 Example: families of the solutions of the Dirichlet problem for the Monge-Ampère equation

From now on D is an open strictly convex and bounded subset of $\mathbb{R}^k$ and we shall consider the solutions of the Dirichlet problem for the Monge-Ampère equation.

Theorem 3.1. *Let* $\mathcal{M} \subset b\mathcal{M}_+(D)$ *and* $\mu_0 \in b\mathcal{M}_+(D)$ *be such that for every* $\mu \in \mathcal{M}$, $\mu \leq \mu_0$ *For each* $\varphi \in C(\partial D)$ *the set* $\{M(\mu : \varphi) : \mu \in \mathcal{M}\}$ *is relatively compact in the topology of the uniform convergence on the set* $\overline{D}$.

Theorem 3.2. *If* $\mathcal{F} \subset C(\partial D)$ *is a bounded set in* $(C(\partial D), \|\cdot\|_\infty)$ *and* $\mu \in b\mathcal{M}_+(D)$, *then* $\{M(\mu;\varphi) : \mu \in \mathcal{F}\}$ *is relatively compact in* $C(D)$ *with respect to the topology of compact convergence.*

Proposition 3.3. *Let* $\mathcal{F} \subset C(\partial D)$ *be such that for each* $\varphi \in \mathcal{F}$, $|\varphi| \leq \varphi_0$, *where* $\varphi_0 \in C(\partial D)$, *and* $\mathcal{M} \subset b\mathcal{M}_+(D)$ *be such that for each* $\mu \in M$, $\mu \leq \mu_0$, *where* $\mu_0 \in b\mathcal{M}_+(D)$. *The set* $\{M(\mu;\varphi) : \mu \in \mathcal{M}$ *and* $\varphi \in \mathcal{F}\}$ *is relatively compact in* $C(D)$ *with respect to the topology of the compact convergence.*

Corollary 3.4. *Suppose that* $(\mu_n)_{n\in\mathbb{N}^\star}$, μ, γ *are positive Radon bounded measures on* D *such that* $(\mu_n)_n \to \mu$ *in the vague topology and* $\mu_n \leq \gamma$ *for all* $n \in \mathbb{N}^\star$. *For each* $\varphi \in C(\partial D)$ *it follows that* $(M(\mu_n;\varphi))_n \to M(\mu;\varphi)$ *uniformly on* $\overline{D}$.

Corollary 3.5. *Let* $(f_n)_n \subset \mathcal{L}_+^\infty(D)$ *be bounded with respect* $\|\cdot\|_\infty$ *and* $f \in \mathcal{L}_+^\infty(D)$ *be such that* $(f_n)_n \to f$ λ *a.e. on* D. *For each* $\varphi \in C(\partial D)$

$$(M(f_n \cdot \lambda;\varphi))_n \to M(f\cdot\lambda;\varphi) \text{ uniformly on } \overline{D}.$$

Proposition 3.6. *Let* $(\mu_n)_n$, μ *be positive bounded Radon measures such that* $(\mu_n)_n \to \mu$ *strongly on the space* $C_c(D)$ *and let* $(\varphi_n)_n \subset C(\partial D)$ *be such that* $(\varphi_n)_n \to \varphi$ *uniformly on* ∂D. *It follows that* $(M(\mu_n;\varphi_n))_n \to M(\mu;\varphi)$ *uniformly on* $\overline{D}$.

Corollary 3.7. *Suppose* $(f_n)_n \subset \mathcal{L}_+^\infty(D)$ *bounded and* λ *a.e. convergent on* D *to the map* f. *If* $(\varphi_n)_n \subset C(\partial D)$ *is uniformly convergent to the function* φ *on* ∂D, *then* $M(f_n\cdot\lambda;\varphi_n))_n$ *is uniformly convergent to the map* $M(f\cdot\lambda;\varphi)$ *on the set* $\overline{D}$.

References

[1] Bertin, E.M.J., *Fonctions convexes et théorie du potentiel.* Preprint nr.89, Univ. Utrecht, 1978.
[2] Bertin, E.M.J., *Convex Potential Theory.* Preprint nr.489, Univ. Utrecht, 1987.
[3] Bourbaki,N., *Élémentes de mathématiques. Integration.* Ch. 1-4, Hermann, Paris, 1969.
[4] Bourbaki,N., *Élémentes de mathématiques. Integration.* Ch. 5, Hermann, Paris, 1956.
[5] Bourbaki,N., *Élémentes de mathématiques. Topologie generale.* Ch. 10, Hermann, Paris, 1961.
[6] Gool, Frans van, *Topics in Non-linear Potential Theory.* Ph.D.thesis, Utrecht, 1992.
[7] Peressini, A.L., *Ordered topological vector spaces.* New York, 1967.
[8] Rockafellar, T., *Convex Analysis.* Princeton Univ. Press, Princeton, 1970.

[9] Schaefer, H.H., *Topological vector spaces.* MacMillan, New York, 1966.
[10] Udrea,C., *Nonlinear Resolvents.* Rev. Roumaine Math. Pures et Appl. **XL**, *7-8*, 691-712, 1995.
[11] Udrea, C., *The Perron-Wiener-Brelot Method and the Monge-Ampère Equation*, Proc. of XVI-th R. Nevanlinna Coll., Walter de Gruyters & Co, 279-288, 1996.
[12] Udrea,C., *On Nonlinear Operators: Boundedness and Maximum Principles.* Rev. Roumaine Math. Pures Appl., **XLVI**, *1*, 125-136, 2001.
[13] Udrea,C., *A Family of Nonlinear Resolvents.*, SEAMS Bull.Math., 29, 785-806, 2005.
[14] Udrea,C., *Resolvents and Nonlinear Potential Theory.*, New Trends in Potential Theory, Theta, 163-173, 2005.

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