PATH COALGEBRA *kQ* AND RIGHT COMODULES OVER *kQ*

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Abstract: *This paper is an introduction in the theory of the category of right comodules over the path coalgebra kQ. Remind that the presentation is a didactic one. So we start with the construction of path coalgebra and then, using coalgebra representations, we find an equivalence from the category of rational representations of Q to the category of right kQ – comodules. Finally, we characterize a localized subcoalgebra of kQ.*

Keywords: path coalgebra, right comodules over kQ, abelian category

1 Preliminary notions.

Definition 1.1. Let k be a field. A k – *coalgebra* is a k – vector space C endowed with two linear applications: $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ such that $(\Delta \otimes I) \circ \Delta =(I \otimes \Delta) \circ \Delta$ and $(I \otimes \varepsilon) \circ \Delta =(\varepsilon \otimes I) \circ \Delta$. We denote a coalgebra C by (C, Δ, ε).

Definition 1.2. Let (C, Δ, ε) be a coalgebra over k. A k – vector space M together a linear map $\rho : M \to M \otimes C$ is called *a right* C – *comodule* if $(I_M \otimes \Delta) \circ \rho = (\rho \otimes I_C) \circ \rho$ and $(I_M \otimes \varepsilon) \circ \rho = i$, where $i : M \to M \otimes k$ is the canonical isomorphism. In the same way we can define a left C – comodule N with the structure map $\lambda : N \to C \otimes N$ (see [3])

We note with $\mathrm{M}^C$the category of right C – comodules and with ${}^C\mathrm{M}$ the category of left C – comodules (much more details in [3]).

Definition 1.3. A *quiver* is an oriented graph $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices and $Q_1$is the set of arrows.

Let $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ where $s(\alpha) = i$ and $t(\alpha) = j$, for every arrow $\alpha : i \to j$ from the vertex i to j . A *path* p in Q is a sequence $p = \alpha_n ... \alpha_1$ in such way that $t(\alpha_i) = s(\alpha_{i+1})$, $i = 1, ..., n-1$.

We denote with $\mathbf{P}$ the set of all paths in Q and for every $i \in Q_0$, the set of all paths starting from i with $\mathbf{P}(i,?)$. A trivial path in Q, denoted by e_i, is a path with the property $t(e_i) = s(e_i) = i$. For every nontrivial path $p = \alpha_n ... \alpha_1$ we define $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$.

A nontrivial path is called *an oriented cycle* if $s(p) = t(p)$.

The length of a path p, denoted by$|p|$, is the number of arrows which compose it. For completeness we consider vertices as trivial paths or paths of length zero. The concatenations of paths: for a path α from i to j and another path β from j to l , their product or concatenation is the path from i to l denoted by $\beta\alpha$.

Example 1.1. Let $(P, \le)$ be a partial ordered set (poset for short). Suppose that P is local finite, means that for every elements $x, y \in P$ such that $x \le y$ in P, the set $[x, y] = \{z \in P \,|x \le z \le y\}$ is finite.

Starting with the poset $(P, \le)$ we construct the quiver $Q = (Q_0, Q_1)$ in the following way:

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- $Q_0 = P$, and for every $x, y \in P$ let $\alpha : x \to y$, $\alpha(x) = \begin{cases} x \to y \,,\, x \leq y \\ 0 \,,\, else \end{cases}$. It means that exist an arrow from x to y only if $x \leq y$ in P;

- Q_1 is the set of arrows between vertices of Q_0.

Next, let $Q = (Q_0, Q_1)$ be a quiver. We can construct a $k -$ vector space, denoted kQ, over k of base Q. We obtain $kQ = \{\sum_{i=1}^{n} a_i p_i \,|a_i \in k,\, p_i\, path\ in\ Q \,,\, n \in N^*\}$.

On this vector space we define a coalgebra structure:

$$\Delta : kQ \to kQ \otimes kQ, \Delta(p) = \sum_{p=p_1 p_2} p_1 \otimes p_2$$

$$\varepsilon : kQ \to k, \varepsilon(p) = \delta_{|p|,0}$$

where if $p = \alpha_t ... \alpha_{s+1} \alpha_s ... \alpha_1$, then $p_1 = \alpha_t ... \alpha_{s+1}$ and $p_2 = \alpha_s ... \alpha_1$, for $1 \leq s \leq t$ and where $|p| = t$ is the length of the path p.

The triplet $(kQ, \Delta, \varepsilon)$ is a coalgebra (the proof is in [5]-[6]) and it is called *the path coalgebra associated to the quiver* $Q = (Q_0, Q_1)$).

Definition 1.4. Let k be a commutative field.

A representation of a quiver $Q = (Q_0, Q_1)$consist in:

1. to associate to every vertex from $Q_0$and every arrow from Q_1, a linear application from the vector space associated to the origin of the arrow considered, to the vector space associated to the vertex of the same arrow.

More precisely, *a representation* V of $Q = (Q_0, Q_1)$ is a set (collection)

$$\{V_i \,|i \in Q_0\}$$

of $k -$ vector spaces of finite dimension together with a set (collection)

$$\{V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)} \,|\alpha \in Q_1\}$$

of $k -$ linear applications.

The *dimension* of V is the map $d_V : Q_0 \to Z_{\geq 0}$, $d_V(i) = \dim V_i \,,\, \forall i \in Q_0$.

If V and W are two representations of the same quiver Q , then a map $\psi : V \to W$ is a set of $k -$ linear applications

$$\{\psi_i : V_i \to W_i \,|i \in Q_0\}$$

such that

$$W_\alpha \psi_{s(\alpha)} = \psi_{t(\alpha)} V_\alpha \,,\, \forall \alpha \in Q_1.$$

By composing of maps V_α, we obtain a linear map V_p which corresponds to a nontrivial path p.

The category of the representations of a quiver Q we denote by $rep(Q)$. Next we present in which way this category is equivalent to the category of comodules over path coalgebra kQ.

For a representation $V = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q and for a path $p \in \mathbf{P}$, the linear application V_p is:

$$V_p = \begin{cases} I_{V_i} \,,\ if\ p = e_i\, for\, some\ i \in Q_0 \\ V_\alpha \,,\ if\ p = \alpha\, for\, some\ \alpha \in Q_1 \\ V_{\alpha_n} V_{\alpha_{n-1}} ... V_{\alpha_1} \,,\ if\ p = \alpha_n \alpha_{n-1} ... \alpha_1 \,,\, where\, \alpha_i\, are\, arrows \,,\, i = \overline{1, n} \end{cases}$$

Definition 1.5. A representation V of the quiver Q is called *rational* if for every $i \in Q_0$ and for every $v \in V_i$, the set $\{p \in P(i,?)\,|V_p(v) \neq 0\}$ is finite.

Examples of representations 1.2.

1. A representations of the quiver

   associated to the poset $(P = \{1,2\}, \leq)$, is a collection of two vector spaces of finite dimention V_1, V_2 together a linear application $V_a : V_1 \to V_2$.

2. A representation of Jordan quiver

   associated to the poset $(P = \{1\}, =)$ is a vector space V_1 together a map $V_a : V_1 \to V_1$.

3. A representation of star-quiver

is a collection of vector spaces $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ together with five linear applications $\left\{V_{a_i} : V_i \to V_6 \,\middle|\, i = \overline{1,5}\right\}$. If all these applications are injective ones we can consider that this representation is in fact a representation of vector subspaces.

2 Path coalgebra kQ and right comodules over kQ

Let V be a rational representation of Q. We define on the vector space $M := \underset{i\in Q_0}{\oplus} V_i$ a structure of kQ – right comodule by Δ_M:

$\Delta_M(v) = \sum_{p\in P(i,?)} V_p(v) \otimes p$, for every $i \in Q_0$ and $v \in V_i$.

Because the representation is rational, the sum above is finite.

Proposition 2.1. Let M be a k – vector space and $\phi : P \times M \to M$ a linear application in the second argument. Let $\phi_p(m) = \phi(p,m)$, for any $(p,m) \in P \times M$. We define the map $\Delta_M : M \to M \otimes kQ$, by $\Delta_M(m) = \sum_{p\in P} \phi_p(m) \otimes p$, for $m \in M$. We obtain that the pair (M, Δ_M) is a right kQ – comodule only if the following conditions are satisfied:

(1) For every $m \in M$ and every $p_1, p_2 \in P$,

$\phi_{p_2}(\phi_{p_1}(m)) = \begin{cases} \phi_{p_2p_1}(m)\ ,\ if\ p_2p_1 \neq 0 \\ 0\ ,\ if\ p_2p_1 = 0 \end{cases}$ and

(2) For every $m \in M$, $m = \sum_{i\in Q_0} \phi_{e_i}(m)$.

Proof. It is enough to verify that the condition (1) is equivalent to $(I_M \otimes \Delta') \circ \Delta_M = (\Delta_M \otimes I_{kQ}) \circ \Delta_M$ and, also, that the (2) condition is equivalent to $(I_M \otimes \varepsilon') \circ \Delta_M = i_M$, where $i_M$is the canonical isomorphism $i_M : M \to M \otimes k$.

(M, Δ_M) is a right kQ – comodule if the diagrams from the definition 2 are commutative.

So, for every m from M , we have

$$((\Delta_M \otimes I_{kQ}) \circ \Delta_M)(m) = (\Delta_M \otimes I_{kQ})(\Delta_M(m)) =$$

$$= (\Delta_M \otimes I_{kQ})\left(\sum_{p_2\in P} \phi_{p_2}(m) \otimes p_2\right) = \sum_{p_2\in P} \Delta_M(\phi_{p_2}(m)) \otimes p_2 =$$

$$= \sum_{p_2\in P}\left(\sum_{p_1\in P} \phi_{p_1}(\phi_{p_2}(m)) \otimes p_1\right) \otimes p_2$$

and also,

$$((I_M \otimes \Delta') \circ \Delta_M)(m) = (I_M \otimes \Delta')(\Delta_M(m)) = (I_M \otimes \Delta')\left(\sum_{p\in P} \phi_p(m) \otimes p\right) =$$

$$= \sum_{p \in P} \phi_p(m) \otimes \Delta'(p) = \sum_{p \in P} \phi_p(m) \otimes \left(\sum_{p=p_1 p_2} p_1 \otimes p_2 \right) = \sum_{p \in P} \sum_{p=p_1 p_2} \phi_p(m) \otimes p_1 \otimes p_2.$$

From above we obtain $(I_M \otimes \Delta') \circ \Delta_M(m) = (\Delta_M \otimes I_{kQ}) \circ \Delta_M(m)$, for every m from U, then (1) is true.

Much more, $((I_M \otimes \varepsilon') \circ \Delta_M)(m) = i_M(m), \forall m \in M \Leftrightarrow m = ((I_M \otimes \varepsilon') \circ \Delta_M)(m) =$

$$= (I_M \otimes \varepsilon') \left(\sum_{p \in P} \phi_p(m) \otimes p \right) = \sum_{p \in P} \phi_p(m) \otimes \varepsilon'(p) = \sum_{p \in P} \phi_p(m) \otimes \delta_{|p|,0} = \sum_{i \in Q_0} \phi_{e_i}(m) \otimes 1 =$$

$= \sum_{i \in Q_0} \phi_{e_i}(m)$, then (2) in true.

Theorem 2.1. The pair $\left(M := \underset{i \in Q_0}{\oplus} V_i, \Delta_M \right)$ is a right kQ – comodule.

The proof result from the above proposition.

Let $\Phi(V) = (M, \Delta_M)$, the functor $\Phi : Rat - rep(Q) \to Mod^{kQ}$ from the category of rational representations of the quiver Q to the category of right kQ – comodules.

Conversely, let kQ_0 be the vector space of base $\{e_i \,|i \in Q_0\}$. We define a k – linear application $\pi : kQ \to kQ_0$ by $\pi(p) = \varepsilon'(p) e_{t(p)}$, for every path $p \in$**P**. Let consider now a right kQ – comodule $M = (M, \Delta_M)$. We define a representation $V = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q , in the following way:

1. For every $i \in Q_0$, let $V_i = \{m \in M \,|(I \otimes \pi) \circ \Delta_M(m) = m \otimes e_i\}$, which is naturally a k vector space because of the linearity of the maps which appear in V_i;

2. For every $m \in M$ we can write in a unique way:

$$\Delta_M(m) = \sum_{p \in P} \phi_p(m) \otimes p,$$

where $\phi_p(m) \in M$, because **P** is a base of kQ.

The uniqueness of $\phi_p(m)$, $p \in$**P** make possible the definition of the map $V_\alpha : V_i \to V_j$ by $V_\alpha(m) = \phi_\alpha(m)$, $m \in V_i$ where $\forall \alpha : i \to j$ is an arrow from Q. Also, the uniqueness of $\phi_p(m)$ demonstrate the linearity of V_α.

Now let's prove that $\phi_\alpha(V_i) \subseteq V_j$.

Proposition 2.2. For every $i \in Q_0$ the following are true:

$$(1) V_i = \{m \in M \,|m = \phi_{e_i}(m)\}; \tag{1}$$

(2) $\phi_\alpha(V_i) \subseteq V_j$, where $\alpha : i \to j$ is arrow in Q;
(3) $\phi_\mu(m) = 0$ only if only $\mu \in$**P**$(i, ?)$;
(4) $V_{e_i}(m) = \phi_{e_i}(m)$, then $m = \phi_{e_i}(m)$;
(5) If $\mu = \alpha_n \alpha_{n-1} ... \alpha_1$ for some arrows$\alpha_1, \alpha_2, ..., \alpha_n$ $(n > 0)$, then $V_\mu(m) = \phi_\mu(m)$ and

$$V_{\alpha_n} V_{\alpha_{n-1}} ... V_{\alpha_1}(m) = \phi_\mu(m).$$

Proof. (1) For every $m \in M$ we have:

$$(I \otimes \pi) \Delta_M(m) = \sum_{\mu \in P} \phi_\mu(m) \otimes \pi(\mu) = \sum_{j \in Q_0} \phi_{e_i}(m) \otimes e_j.$$

Then, $V_i = \{m \in M \,|\phi_{e_j}(m) = \delta_{ij} m\} \subseteq \{m \in M \,|m = \phi_{e_i}(m)\}$.
(2) For every $m \in V_j$, we have $\phi_{e_j}(\phi_\alpha(m)) = \phi_{e_j \alpha}(m) = \phi_\alpha(m)$.
We obtain $\phi_\alpha(m) \in V_j$.
The relations (3), (4) and (5) are immediately.

Proposition 2.2. Let V be a representation of Q obtained from a right kQ – comodule M as above. Then the following are true:

1. V is a rational representation.
2. $M = \underset{i \in Q_0}{\oplus} V_i$, as a sum of vector spaces.
3. For every $i \in Q_0$ and $v \in V_i$ we have

$$\Delta_M(v) = \sum_{\mu \in P(i,?)} V_\mu(v) \otimes \mu.$$

The proof follows immediately from the above lemmas.

If we denote with $\Psi(M) := V$, we obtain a functor $\Psi : Mod^{kQ} \to Rat - rep(Q)$ which is equivalent and quasi-inverse to $\Phi : Rat - rep(Q) \to Mod^{kQ}$, means that the category of rational representations of the quiver Q is equivalent to the category of right kQ – comodules.

Example 2.1.

Let Q be the quiver $(\{1,2\}, \{\alpha_i \,|i \in \mathrm{N}\}, t, s)$ where $s(\alpha_i) = 1$ and $t(\alpha_i) = 2$ for every $i \in \mathrm{N}$. Now, we consider the representation V with $V_1 = k = V_2$ and $V_{\alpha_i} = Id$, $\forall i \in \mathrm{N}$. Then V is 2-dimensional, but is not rational. We observe that any representation of the sub-quiver $Q^{(n)} := (\{1,2\}, \{\alpha_i \,|i = 1, ..., n\}, t, s)$, $n \in N$ fixed, of Q is rational, then they are considered as kQ – comodules.

3 Localization in the path coalgebra *kQ*

3.1 Some preliminaries.

In the following paragraph we consider that the reader is familiarized with notions from the theory of categories (see also [6]). Recall (from [4]-[5]) that if C is an abelian category, then a subcategory A of C is dense if and only if from every exact sequence $0 \to X' \to X \to X'' \to 0$ of objects from C, we have that $X \in$A if and only if X' and X'' are from A. For every dense subcategory A of C exist an abelian category C /A and an exact functor $T : \mathrm{C} \to \mathrm{C}/\mathrm{A}$ such that $T(X) = 0$, $\forall X \in$A and with the universal property: for every functor $H : \mathrm{C} \to \mathrm{C}'$ such that $H(X) = 0$, $\forall X \in$A, does exist an unique functor $\overline{H} : \mathrm{C}/\mathrm{A} \to \mathrm{C}'$ such that $\overline{H} \circ T = H$. The category C /A is called the factorization category of C with respect to the subcategory A.

Also, a dense subcategory A of C is called localizing if the functor $T : \mathrm{C} \to \mathrm{C}/\mathrm{A}$ has a right adjunct $S : \mathrm{C}/\mathrm{A} \to \mathrm{C}$, in this case S is called the section functor of T .

In the particular case in which C is a Grothendieck category (ex. the category of right comodules of a coalgebra C, M^C), a dense subcategory A of C is localizing if and only if it is closed under direct sums.

Let C and D be two coalgebras and $M \in \mathrm{M}^C$ with the structure map $\rho_M : M \to M \otimes C$ and $N \in {}^C\mathrm{M}$ with the structure map $\lambda_N : N \to C \otimes N$. The cotensor product $M\ ?_C N$ is the kernel of the linear application

$$\rho_M \otimes N - M \otimes \lambda_N : M \otimes N \to M \otimes C \otimes N.$$

In [1] is given the next theorem: If A is a localizing subcategory of $M^C$and X is a injectve quasi-finite right C comodule such that $A = A_X$ and if we consider the injective Morita-Takeuchi context (D, C, X, Y, f, g) defined by X as in [1], then the functors

$T = (\text{-})?_C Y : M^C \rightarrow M^D$ and $S = (\text{-})?_D X : M^D \rightarrow M^C$

define a localization of M^C with respect to the localizing subcategory A. In particular $M^C/$ A is equivalent to M^D.

3.2 Localization in the path coalgebra kQ

Now let $e \in kQ^*$ be an idempotent element. Then $e(kQ)e$ can be endowed with a coalgebra structure given by:

$\Delta_{e(kQ)e}(epe) = \sum ep_1e \otimes ep_2e$ and $\varepsilon_{e(kQ)e}(epe) = e(p)$

where $\Delta(p) = \sum_{p=p_1p_2} p_1 \otimes p_2$, for every $p \in kQ$.

Since $(kQ)e$ is a quasi-finite injective right kQ – comodule, we can consider the injective Morita-Takeuchi context $(e(kQ)e, kQ, (kQ)e, e(kQ), f, g)$ as in [1], then the functors

$T = (\text{-})?_{kQ}e(kQ) : M^{kQ} \rightarrow M^{e(kQ)e}$ and $S = (\text{-})?_{e(kQ)e}(kQ)e : M^{e(kQ)e} \rightarrow M^{kQ}$

define a localization of M^{kQ} with respect to the localizing subcategory A_e= Ker $T = \{M \in M^{kQ} \ / \ M \ ?_{kQ}e(kQ) = 0 \} = \{ M \in M^{kQ} \ / \ eM = 0\}$.

Much more, for any idempotent $e \in kQ^*$ and any vertex $x \in Q_0$, we have either $e(x) = 0$ or $e(x) = 1$. But two idempotent elements $e, f \in kQ^*$ are equivalent if and only if $e_{|Q_0} = f_{|Q_0}$, and so we have that every localizing subcategory of $M^{kQ}$is associated to an idempotent element $e \in kQ^*$ such that $e(p) = 0$ for any path p with length $|p| > 0$.

In [1], for any such idempotent element e it is defined a new quiver Q^e as:

- $Q_0^e = \{x \in Q_0 / e(x) = 1\}$;

- Q_1^e is the set of paths $p = \alpha_n...\alpha_1$ in Q such that $e(s(p)) = e(t(p)) = 1$ and $e(s(\alpha_i)) = 0, \forall i = \overline{2, n}$.

Theorem 3.1. The localized coalgebra $e(kQ)e$ is isomorphic with the path coalgebra of the quiver Q^e.

Consequence 3.2. The functor $T = (\text{-})?_{kQ}e(kQ) : M^{kQ} \rightarrow M^{e(kQ)e}$can be regarded as a functor from $Rat - rep(Q)$ to $Rat - rep(Q^e)$.

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