CONSIDERATIONS ON THE STABILIZED EQUILIBRIUM OF THE HOT PLASMA

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Abstract: *We introduce a variational problem on a Sobolev space associated with an eigenvalue problem for a magnetic flux, characterizing the quasi-static equilibrium of plasma. We obtain the existence and regularity results of a weak solution. Topological arguments – topological degree, ensure the control of a unique solution depending on a parameter – magnetic permeability of a total ionized gas and on the eigenvalues of an associated Dirichlet problem. The control on a technological parameter to the process initiation ensures equilibrium stabilization and controlled functionality of the machine.*

Keywords: *equilibrium, plasma, plasmatron*

1. INTRODUCTION

We have in view a technological process of producing a plasma jet inside a burner and confine it on the median axis of the burning chamber in magnetic field. The process is described by a nonlinear mathematical model of quasi-static equilibrium of the jet, it is introduced a state function of metallic plasma-vacuum-chamber system. We consider the system state function to characterize the domain stability occupied by the ionized gas jet. We will indicate the magnitude and the direction of the magnetic field induction, applied in addition to confine the jet on the median direction of the burner chamber towards the ejection hole.

The equations that we study, in a general context of the two dimensional Euclidean space, particularize the MHD equations of a process of producing and stabilization of the plasma in magnetic field in a tri dimensional chamber.

A technological parameter, which signification is given by the magnetic permeability of total ionized gas, allows us the statement of an eigenvalue problem for an elliptic operator, defined by a potential magnetic flux. The variational formulation on a *Sobolev* space of the nonlinear problem, which governs the phenomenon and topological arguments ensure the existence and the smoothness of the solution, for general problems see [3], [6], [12], [17]. In the case of satisfying some quasi linear conditions on the potential, a uniqueness result is derived. For similar problems we refer to [1], [5], [10]. Other results on the solution existence for some problems in plasma physics are found in [2], [9], [11], [13], [19], on regularity results of the solution in [8], [18], on non uniqueness in [14], [15], [16], also on uniqueness in [4].

The structure of this paper is organized as follows: in the Section 2 it is introduced a technological model of a *plasmatron*, in Section 3 is formulated the generalized nonlinear problem of the quasi-static equilibrium of plasma jet and in Section 4 are given the conditions for the general state function and are derived the existence and regularity results. The hypothesis of a polynomial asymptotic behavior permit us the fruitful considerations. Some technological consequences are deduced.

2. STATEMENT OF THE TECHNOLOGICAL MODEL

The technological premises adapted to a stability study of plasma jet**,** for an industrial elaboration process of the steel in an electric arc furnace. The heat exchange is made by the *Joule* effect, less through radiation and convection, which creates a deficiency in the leadership of the process, through uneven heating in areas of interior surface, with consequences in refracting material erosion of the wall. The plasma burner is a device which produce a hot gas, totally ionized, the jet geometry depending on the electric potential, on the current discharge, on the pressure and the working gas flux, the discharge chamber geometry, the size of ejection hole. Simplistically, a burner has the following components: the burner head, the isolator and the burner body with electrode, see Fig. 1.

The head and the body of the *plasmatron*, see A. *Arabagiu* [1], are made of cupper with cooling chamber and the electrode is made of wolfram. It is applied a potential difference by connecting the burner body to the cathode and the head to the anode, so that between the electrode and the burner head it is formed an electric arc. The heat in the arc is transferred to the plasma gas and the totally ionized gas jet is stabilized in the auxiliary magnetic field.

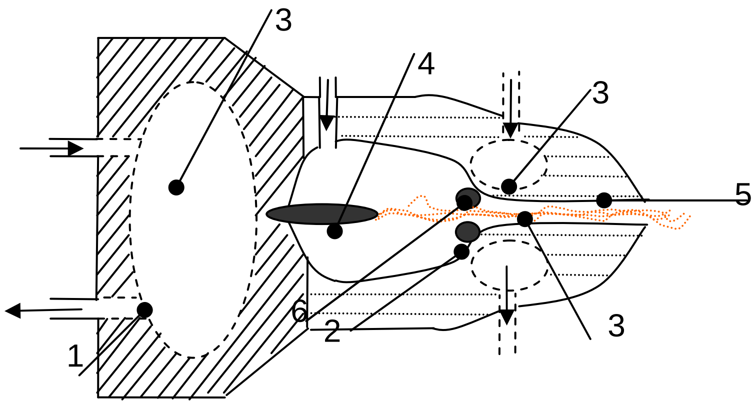

Fig. 1.
Schema of Plasmatron

1- body, 2- head, 3- cooling chamber, 4- electrode, 5- jet 6 anodic ring.

This *plasmatron* can be used in the technology of metal melting ensuring the heat transfer from the plasma jet to the metallic load by radiation and a small part dissipated in the anode area of the equipment by conduction. For a homogeneous irradiation of the metallic bath, are placed three *plasmatrons* at 120^0 in the ceiling of the electric furnace, each of them producing a jet of 500 to 1400 mm length, which completely covers by spreading the working area. The control of the functioning process, with the possibility of changing the power level, without the interruption danger, due to the confined plasma flow self maintained, is governed by the modification of the radiation parameter while the metal is melted.

From the theoretical point of view, we are interested in the stability of the ionized gas jet on the median line of the chamber, controlling a magnetic field potential. Mathematically, equi-potential surfaces of the magnetic induction field $\vec{B}$, include the free boundary surface of the plasma jet and also the chamber surface, which allows the control of the jet confinement.

There are made simplified hypotheses on the phenomenon occurrence: the inertial forces, the hydrodynamic pressure of the ionized gas are neglected in contrast to the electromagnetic pressure p, which develops in plasma jet.

It is considered the electromagnetic forces field, perpendicular to the magnetic induction field $\vec{B}$ and to the electric current $\vec{j}$, $\vec{f} = gradp = \frac{1}{C}\vec{j}\times\vec{B}$.

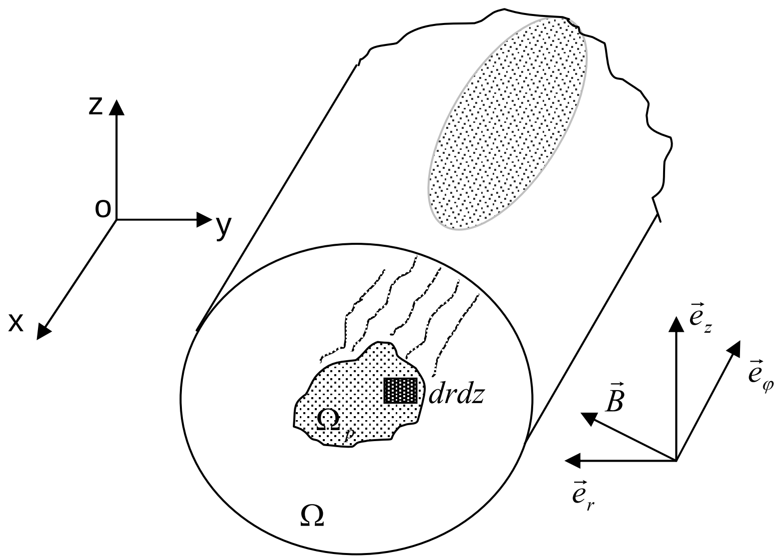

Fig. 2. Cartesian system of reference placed locally at the conjunction of the chamber with the ejection hole.

We argue the choice of such Cartesian system of reference (Fig. 2.): an electromagnetic force variation from the surface $p = ct.$ to the surface $p + dp = ct.$ indicates the direction $\vec{e}_r$, that is $\vec{e}_r = \frac{gradp}{\|gradp\|}$, as direction of the force field. The direction of the magnetic induction field $\vec{B}$ defines $\vec{e}_z$ and $\vec{e}_\varphi = \vec{e}_r \times \vec{e}_z$. In our reference system, the process of production, of stability and plasma flow present a translation invariant along the plasma jet. We keep in mind the equations of the electromagnetic field disturbed by plasma,

$$-gradp + \frac{1}{4\pi} rot\vec{B}\times\vec{B} = 0,\ div\vec{B} = 0,\ rot\vec{B} = \frac{4\pi}{C}\vec{j},\ div\vec{j} = 0.$$

The equi-potential surface, on which the field $\vec{B}$ is tangent, has the equation

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}.$$

We realize the vector product of the two fields $\vec{f}$ and $\vec{B}$, so we have $\vec{B}\times\vec{f} = \vec{B}\times gradp = \frac{1}{C}\vec{B}\times\left(\vec{j}\times\vec{B}\right) = \frac{1}{C}\left(-\left(\vec{B}.\vec{j}\right)\vec{B} + \left|\vec{B}\right|^2\vec{j}\right)$, from here we deduce

$$\vec{j} = \frac{C}{B^2}\left(\vec{B}\times gradp\right) + \frac{1}{B^2}\left(\vec{B}.\vec{j}\right)\vec{B} = \frac{C}{B^2}\left(\vec{B}\times gradp\right) + \frac{1}{4\pi}\left(\vec{B}.rot\vec{B}\right)\vec{B},$$

expression which will be used for the calculation of the total current which crosses Ω. We remember that $\vec{B}$ is a field in the plan $(\vec{e}_r, \vec{e}_z)$ and the plan $(\vec{e}_r, \vec{e}_z)$ determines the transversal section Ω into the fluid jet. The position of the field $\vec{B}$ referred to the system $(\vec{e}_r, \vec{e}_\varphi, \vec{e}_z)$ leads to a relation like: $\vec{B}.(d\vec{e}_r \times d\vec{e}_z) = 0$, with the dot we denote the scalar product. In contrast with this, for the product of the field $\vec{j}$ with $d\vec{e}_r \times d\vec{e}_z$, we have

$$\vec{j}.(d\vec{e}_r \times d\vec{e}_z) = \frac{C}{B^2}(\vec{B} \times gradp).(d\vec{e}_r \times d\vec{e}_z) = \frac{C}{B^2}\begin{vmatrix} \vec{B}.d\vec{e}_r & \vec{B}.d\vec{e}_z \\ gradp.d\vec{e}_r & gradp.d\vec{e}_z \end{vmatrix} =$$

$$= \frac{C}{B^2}\begin{vmatrix} 0 & Bdz \\ gradpdr & 0 \end{vmatrix} = -\frac{C}{B} gradp(r,z)drdz \,.$$

When crossing the surface Ω, the current density in the domain between the equipotential surfaces $p = ct., p + dp = ct.$ produces a quantity of current

$$dI = \iint_\Omega \vec{j} d\vec{e}_r \times d\vec{e}_z = -C\iint_\Omega \frac{1}{B} gradpdrdz \text{ (but } gradpdr = dp \text{) } = -Cdp\int_\gamma \frac{1}{B}dz < 0,$$

where γ is a loop of a force line.

We admit as a working hypotheses: the specific flux volume of magnetic induction between two equi-potential surfaces is constant, than $U = \int_\gamma \frac{1}{B}dz = = \iiint_V \frac{1}{(\vec{B}, d\vec{e}_r, d\vec{e}_\varphi)} dV$ is constant along the equi-potential surfaces of the magnetic field; we can take as a parametric representation of the equi-potential magnetic surfaces the equation $U(x,y,z) = ct.$ A consequence of this hypotheses is just the unchanged total current which crosses Ω_p, $dI = \iint_{\Omega_p} j_\varphi drdz$, where $d\vec{S} = d\vec{e}_r \times d\vec{e}_z$, $dS = drdz$ is an area element on the section surface Ω.

3. AN EQUILIBRIUM GENERALIZED PROBLEM

We realize an abstract setting of the equilibrium problem for the confined plasma in the *Hilbert* space. For the study of the week solutions as critical points, for an operational equation we realize the operational writing, defined in a Banach space, equivalent to a variational problem.

Let $\Omega \subset R^n$, be a bounded and measurable sub domain, $\Gamma = \partial\Omega$ its regular boundary; we introduce *Sobolev* spaces $W^{2,p}(\Omega) = \{u \in L^2(\Omega) / D^\alpha u \in L^p(\Omega), |\alpha| \le 2\}$, D^α is the derivate along the space $D'(\Omega)$ (of the distributions), with the norm $\|u\|_{2,p} = \left(\sum_{|\alpha|\le 2} \int_\Omega |D^\alpha u(x)|^p dx \right)^p$,

the Hölder function space

$$C^{1+\alpha}(\Omega) = \left\{ u \in C^1(\Omega) / \frac{|D^\sigma u(x) - D^\sigma u(y)|}{|x-y|^\alpha} < +\infty, |\sigma| = 1, \forall x, y \in \Omega \right\},$$

with the norm $\|u\|_{1,\alpha} = \sum_{|\sigma|=1} \max\left|D^\sigma u(x)\right| + \max_{|\sigma|=1} \frac{\left|D^\sigma u(x) - D^\sigma u(y)\right|}{|x-y|^\alpha}$, the differential operator L

$$: H^1(\Omega) := W^{2,1}(\Omega) \to \ L^2(\Omega),\ \mathrm{L}u(x) = -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + a_0(x)u(x),$$

with a_{ij} derivable functions, with $D^\sigma a_{ij} \in C^{1+\alpha}(\Omega)$, symmetric, with $|\sigma| = 1$, a_0 a continuous *Hölder* function, L is an elliptical uniform operator, or it exist $\eta \in R_+$, so that

$$a_{ij}(x)\xi_i\xi_j + a_0(x)\xi_0^2 \geq \eta\left(|\xi|^2 + \xi_0^2\right),\ (\forall)\xi \in R^n, \xi_0 \in R\ .$$

Remark 3.1. The operator L from $H^1(\Omega)$ to $L^2\Omega$ is a linear operator, bounded and uniform elliptic.

We consider $h : \Omega \times R \to R$, a nonlinear function, satisfying the *Caratheodory* condition, derivable in t, with $\frac{\partial h}{\partial t}(.,.)$ a Lipschitz, function satisfying: $\frac{\partial h}{\partial t}(x,t) > 0$, for $t < 0$ (positively), $\frac{\partial h}{\partial t}(x,t) = 0 \Leftrightarrow t \geq 0$ a.e. on **R** (zero on semi axis R_+), $(\exists)b_1, b_2 \in R_+$,$\beta > 1$, for $n \leq 2$; $\beta = \frac{n}{n-2}$, for $n > 2$, so that $b_1\left(|t|^\beta - 1\right) \leq \frac{\partial h}{\partial t}(x,t) \leq\ b_2\left(|t|^\beta + 1\right)$ (asymptotic behavior).

Remark 3.2. The state function *h* allows the stating of an operator, which under conditions of an asymptotic behavior, is an operator of overlapping $B : H^1(\Omega) \to L^2(\Omega)$, $Bu(x) = \frac{\partial h}{\partial u}(x, u(x)) = F(x, u(x))$. Indeed, we have $H^1(\Omega) \subset L^\delta(\Omega)$, with continuous inclusion, with a relation between parameters : $\frac{1}{\delta} = \frac{1}{2} - \frac{1}{n} = \frac{n-2}{2n}$, and $F(x, u(x)) \in\ L^2(\Omega)$, since $u \in L^\delta(\Omega)$ (even $u \in H^1(\Omega)$).

We introduce the eigenvalue problem,

Problem 3.1. Determine $\Omega_p \subset \Omega$, $u \in H^1(\Omega)$, $\lambda \in R_+$, so that the equation $\mathrm{L}u(x) = \begin{cases} \lambda Bu(x), x \in \Omega_p \\ 0, x \in \Omega - \Omega_p \end{cases}$ and the boundary conditions $u = 0$ on Γ_p, $u = \gamma_u$ on Γ, $I = -\int_{\Gamma_p} \frac{\partial u}{\partial \nu} d\Gamma > 0$ be satisfied, and u and $\frac{\partial u}{\partial \nu}$ are continuous on Γ_p, cu $\frac{\partial .}{\partial \nu}$ co normal derivative associated to the operator L, $\frac{\partial .}{\partial \nu} = a_{ij}\frac{\partial .}{\partial x_i}.\nu_j$.

For treating the *Problem* 3.1 we will realize an operational formulation, equivalent to the variational formulation in *Sobolev* space. We consider the real function $F : R^n \times R \to R$, $F(x;t) = \int_0^t \frac{\partial h}{\partial t}(x;\tau)d\tau$, for whose realization in space $L^2(\Omega)$ we find a functional

$$F_2 : L^2(\Omega) \to R\,,\ F_2(v) = \int_\Omega \{h(x;-u_-(x)) - h(x;0)\}dx\,,$$

which proves to be *Gâteaux* differentiable on space $H^1(\Omega)$, [13], with the differential $(F_2'(u), v) = \int_\Omega \frac{\partial h}{\partial u}(x;-u_-(x))v(x)dx = (B(-u_-), v)$, differential F_2' must be understand from $H^1(\Omega)$ la $H^{-1}(\Omega)$, because $L^\infty(\Omega) \subset H^{-1}(\Omega)$. Functional F_2 is weak continuous on W, indeed, let $\{v_n\}_{n\in N} \subset W$, be weak convergent $v \in W$, immersion theorem [12] for Ω bounded domain, with $\Gamma \in C^1$, ensure $H^1(\Omega) \subset L^p(\Omega)$, with compact inclusion for $1 \le p \le \frac{2n}{n-2}$, which is strong convergence $v_n \to v$ in $L^p(\Omega)$. Since the function h is continuous in the variable t, with the hypothesis of asymptotic comportment, we have $|F(x;u(x))| \le C_1 + C_2|u(x)|^{\beta+1}, \beta = \frac{n}{n-1} < \frac{2n}{n-2}$, it results that *F*, similar to the defined in the bi-dimensional case, has a continuous realization from $L^{\beta+1}(\Omega)$ to $L^1(\Omega)$, in particular from $H^1(\Omega)$ la $L^1(\Omega)$, so that $F(x, v_n(x)) \to F(x, v(x))$ in $L^1(\Omega)$. In consequence $F_2(v_n) \to F_2(v)$. We note the working space in the study of the *Problem* 3.1 with $W = \{v \in H^1(\Omega) / \gamma_0 v = \gamma_v = ct.pe\ \Gamma\}$, having the norm of the space $H^1(\Omega)$, we define the bilinear form $a(.,.)$, bounded, symmetric and differentiable on W, nonlinear form $b(.,.)$,

$$a(u,v) = \frac{1}{2}\int_\Omega \left\{ a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x) + a_0(x)u(x)v(x) \right\} dx\,,\ b(u,v) = \int_\Omega Bu(x)v(x)dx\,,$$

which allows us the writing of the variation equation: determine $(\lambda, u(\lambda)) \in R_* \times W$, so that

$$(PV) \qquad a(u,v) - \gamma_v I = \lambda b(-u_-, v), (\forall) v \in W\,.$$

We consider in the same time the quadratic functional $F_1(v) = \frac{1}{2}a(v,v) - \gamma_v I$, *Fréchet* differentiable, with the differential given by $(F_1'(u), v) = a(u,v) - I\gamma_v$, the product $(.,.)$ being understand in the duality of spaces $H^1(\Omega), H^{-1}(\Omega)$, so it is continuous on *W*. From variational formulation (PV) of the quasi-static equilibrium of the plasma, it is derived a weak formulation of the critical point: determine $\{\lambda, u(\lambda)\} \in R_+ \times W$, so that

$$(PPC) \qquad F_1'(u) = \lambda F_2'(u),\ \text{in } D'(\Omega)\,.$$

Relating to the existence of the critical point *u* and to the corresponding critical value λ to the equation (PPC), we underline the following result (example [12]):

Theorem 3.1. Let S the level subset determined by the functional F_2 on W, $S = \{v \in W / F_2(v) = C = ct.\}$. The element $u \in W$ is a critical point of the problem (PPC), or it exist $\lambda \in R_+$ and $F_2(u) \neq 0$, so that $F_1^{'}(u) \equiv \lambda F_2^{'}(u)$, if and only if u is a point of minim of the functional $F_1(v) = \frac{1}{2} a(v,v) - \gamma_v I$ on the subset S.

4. VARIATIONAL FORMULATION AND THE EXISTENCE RESULTS.

With the help of this result the problem of determining the weak solution would be reduced to $\inf_{v \in S} F_1(v)$.

Theorem 4.1. Functional F_1 is inferior bounded on the surface *S,* for the norm in *W* and it reaches the minimum al least in a point $u \in W$, for which $F_2'(u) \neq 0$.

Proof. The nonlinearity of the two functionals we adopt to the best *Banach,* space then, we have to use the submerging $W \subset L^{\beta+1}(\Omega)$ to extend the functional F_1 to a space which will allow us to use the asymptotic behavior hypothesis of the functional F_2. We make a few estimations in space $L^{\beta+1}(\Omega)$. From the first part of the asymptotic behavior inequality, we deduce $b_1\left(\frac{|u_-|^{\beta+1}}{\beta+1} - |u_-|\right) \leq -\{h(x;-u_-(x)) - h(x;0)\}$, is integrated on the domain Ω, it is considered the level surface equation, it is found $c_3 \int_\Omega |u_-(x)|^{\beta+1} dx \leq C +$ $+ c_4 \int_\Omega |u_-(x)| dx \leq C + c_5 \int_\Omega |u_-(x)|^{\beta+1} dx$, so it exist a superior boundary of the application $u \to u_-$, $\|u_-\|_{\beta+1} \leq c_6 < +\infty$. For checking inferior $L^{\beta+1}(\Omega)$-bounding we realize by logical negation, we admit that exists $\{v_n\}_{n \in N} \subset W$, so that $v_n \in S, n \in N$, and $\|(v_n)_-\|_{\beta+1} \to 0$. We keep in mind the empty values hypothesis on the positive semi axes of the function $\frac{\partial h}{\partial t}$; we make the restriction on the sub domain Ω_p, for which the functions $v_n \in S, n \in N$ can be represented by their negative parts, we obtain $v_n \to 0$, a.e. in Ω_p. From the continuity of the functional F_2 we deduce $F_2(0) = \lim_{n\to\infty} F_2(v_n) = 0 = C \neq 0$, which opposes the condition $C > 0$, consequently it exists $0 < c_7 \leq \|u_-\|_{\beta+1}$. It was used essentially the restriction $u \in S$ for boundedness of the nonlinear application $u \to u_-$, which participates at defining the domain Ω_p. From *Temam* [18] we take a majorating result for the application $u \to u_-$ in $H^1(\Omega)$: for $\alpha > 0$, $\beta > 1$, exist $\delta(\alpha, \beta, \Omega) > 0$, so that $\|u_-\|_{\beta+1} \leq \alpha \|gradu\|^2_{0,2} + \delta(\alpha,\beta,\Omega)\, \|u_-\|_\sigma, \sigma > 1$. We do here $\sigma = \beta + 1$ and we combine with $\|u_-\|_{\beta+1} \leq c_6 < +\infty$, we deduce $\|u_-\|_{\beta+1} \leq \alpha \|gradu\|^2_{0,2} + C(C+1)\widetilde{c}_1$. With this preparing we realize a inferior boundedness of the functional F_1. First we use the ellipticity hypothesis and a *Poincaré* inequality on Ω, we have $F_1(v) \geq \frac{\eta}{2}\|gradv\|^2_{0,2} + I\gamma_v, (\forall) v \in V$, γ_v being the value of the constant of the function v on Γ. We have sufficient regularity for the boundary Γ, we use a result of [12], there exists $\widetilde{c}_2 > 0$, so that $\|\gamma_0\|_{1/2} \le \widetilde{c}_2 \|v\|_{1,2}$. Following we have

$$I_1(v) \ge \frac{\eta}{2}\|gradv\|_{0,2}^2 - I\widetilde{c}_2\left(\|v\|_{0,2}^2 + \|gradv\|_{0,2}^2\right)^{1/2} \text{(we use ,for } \beta+1>2\text{)} \ge$$

$$\frac{\eta}{2}\|gradv\|_{0,2}^2 - I\widetilde{c}_2\|gradv\|_{0,2}^2 - I\widetilde{c}_2\left(\|gradv\|_{0,2} + (C+1)\widetilde{c}_1\right) \ge \left(\frac{\eta}{2} - I\widetilde{c}_2\right)\|gradv\|_{0,2}^2 -$$

$$-I(C+1)\widetilde{c}_1\widetilde{c}_2 .$$

Finally, we obtain the bounded ness $F_1(v) \ge k_1\|gradv\|_{0,2}^2 - k_2$, cu $k_1, k_2 > 0$.

Now, let $\{v_n\}_{n\in N} \subset S$ a minimized sequence on S and let $i = \inf_{n\to\infty}\{F_1(v)/F_2(v) = C > 0\}$. From the two inequalities $F_1(v) \ge k_1\|gradv\|_{0,2}^2 - k_2$, $b_1\left(|t|^\beta - 1\right) \le \frac{\partial h}{\partial t}(x,t)$, it will results that the sequence $\{v_n\}_{n\in N}$ is bounded in space $H^1(\Omega)$, consequently, it exists a subsequence, $\{v_n\}_{n\in N}$, weakly convergent to $u \in H^1(\Omega)$. From compact inclusion $H^1(\Omega) \subset L^2(\Omega)$ it results strongly convergence $v_n \to u$, in $L^2(\Omega)$. The functional F_1 is weakly inferior semi continuous, we have $F_1(u) \le \liminf_{n\to\infty} F_1(v_n) = i$ $= \inf_{v\in S} F_1(v)$, so that *u* is a point of minimum for the functional F_1.

We admit that it would exist $F_2(u) = 0$, which means $\left(F_1'(u), w\right) = 0, (\forall) w \in W$. In particular, for the constant function $w = \gamma \in W$ we have

$$0 = \left(F_2'(u), w\right) = \int_\Omega \frac{\partial h}{\partial u}(x, -u_-(x))w(x)dx = \gamma\int_\Omega \frac{\partial h}{\partial u}(x, -u_-(x))dx = \gamma\int_{\Omega_p} \frac{\partial h}{\partial u}(x, -u_-(x))dx = \gamma I > 0,$$

which is a contradiction. More, if it exist $\frac{\partial h}{\partial u}(x; -u_-(x)) \equiv 0$, from the behavior hypothesis of the state function *h*, we would have $u_- \equiv 0$, (a.e.) in Ω, or $\Omega_p = \emptyset$,

$$F_2(u) = \int_\Omega \{h(x; -u_-(x)) - h(x;0)\}dx = 0 = C \ne 0,$$

again a contradiction. We suppose $F_2(u) = 0$ is false, and the theorem is demonstrated.

How much regularity does the solution of the problem (PPC) has and which is the relation between the weak solution of the variational equation and the classical solution of the eigenvalue problem? We will find in the following result an extension for the similar result from *Temam* [17], for n = 2.

Theorem 4.2. Let $u \in W$ a solution of the problem (PPC), or for the variational equation (PV), then $u \in W^{3,\beta+1}(\Omega)$, $\beta > 1$ and $u \in C^{2+\alpha}$, $0 \le \alpha < 1$. If we denote $\Omega_- = \Omega_p = \{x \in \Omega / u(x) < 0\}$, $\Omega_+ = \Omega - \Omega_p = \{x \in \Omega / u(x) > 0\}$, the domain occupied by the jet, the empty domain, then *u* is the solution of Problem 4.1. The subset $\Gamma_p \cap \Omega$ has empty inside in R^n, and if $a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\vec{e}_i \ne \vec{0}$, in a neighborhood of the point $x \in \Gamma_p$, and then the variety Γ_p is a surface defined by a function of class $C^2(\Omega)$ and $\frac{\partial u}{\partial \nu}$ is continuous on Γ_p. More, if $\Gamma_p \in C^1(\Omega)$, then u is not zero in Ω_p and $I = \int_{\Gamma_p} \frac{\partial u}{\partial \nu}(x)d\Gamma > 0$.

Proof. Let u be a solution of the minimum problem for the functional F_1, then $u \in H^1(\Omega)$ and satisfies the variational equation (PV). The feature of the free boundary problem it is preserves into the nonlinear term $\frac{\partial h}{\partial t}(.,.)$. On consider the space of indefinite differentiable functions $C_\gamma^\infty(\Omega)$, with the constant value on the boundary Γ; from the (PV) taking $v \in C_\gamma^\infty(\Omega)$, we obtain in the distributional sense: $Lu(x) = \frac{\partial h}{\partial t}(x;-u_-(x))$, $(\forall)x \in \Omega$, $u|_\Gamma = \gamma_u$ on Γ, $-u_- \in L^{p+1}(\Omega)$ and $\frac{\partial h}{\partial t}(.,-u_-(.)) \in L^{p+1}(\Omega)$. The theory of the second elliptic equations assure for u a regularity like $W^{2,p+1}(\Omega)$, $p \geq 1$, but from the *Sobolev* theorems it results $u \in C^1(\Omega)$. Moreover, $u \to u_-$ is a Lipschitz function, we have $u_- \in W^{1,p+1}(\Omega)$, $p \geq 1$, then $\frac{\partial h}{\partial t}(.,-u_-(.)) \in W^{-1,p+1}(\Omega)$. We apply iteratively the regularity theorems for the elliptic second order problems, we find $u \in W^{3,p+1}(\Omega) \cap C^{2+\eta}(\Omega), p > 1, 0 \leq \eta < 1$. In $\Omega - \Omega_p$ we have $u > 0$ and $Lu = 0$, $u = \gamma_u$ on Γ and in $\Omega_p : Lu(x) = \frac{\partial h}{\partial t}(x,-u(x)), u = 0$ on Γ_p. From the coefficient regularity results u as an analytic function in $\Omega - \Omega_p$ and if h is a analytic function, then u is analytic too in Ω_p, [7]. From the analytical expression of the S restriction we deduce $u_- \neq 0$, otherwise we have C=0, from which, one can conclude $\Omega_p \neq 0$, hence there exists the plasma occupied domain. We have in mind that $u = \gamma_u > 0$ on Γ, then $\Omega - \Omega_p$ is a non void subset (the plasma does not occupy the entire cavity Ω). If Γ_p has a void interior in R^n then there is analytical u, therefore $Lu = 0$ in $\Omega - \Omega_p$ and u=0 in $\mathrm{Int}(\Gamma_p)$ which is an open subset in R^n, so u is zero on the entire subset $\Omega - \Omega_p$ and this contradicts $\gamma_u \neq 0$ on Γ_p. Let now x, a generic point on Γ_p, for which $a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\vec{e}_i \neq \vec{0}$. From implicit function theorem results that Γ_p is a local regular surface in a vicinity of x and $u|_{\Gamma_p} \in C^2(R^n)$. We determine the flux through the boundary. We multiply the equation in *Problem* 3.1 with $v \in C_0^\infty(O \cap (\Omega - \Omega_p))$ where O is an open subset in R^n, vicinity for $x \in \Gamma_p$.

Because $u \in C^1(\Omega)$, in a local writing we obtain $\int_{\Gamma_p} \frac{\partial u}{\partial \nu} v d\sigma + a(u,v) = \lambda \frac{\partial h}{\partial t}(.,-u_-(.))$ which compared with (PV) leads to $\int_{\Gamma_p} \frac{\partial u}{\partial \nu} v d\sigma = 0$, hence the jump of the function $\frac{\partial u}{\partial \nu}$ to the free boundary is zero for any position of the boundary, that proves the continuity of the function $\frac{\partial u}{\partial \nu}$.

Using the local continuity of the flux $\frac{\partial u}{\partial \nu}$ it verifies the condition that the function u is non zero in the domain Ωp.

5. COMMENTS ON RESULTS AND CONCLUSIONS

As particular quasi-static equilibrium problem we are able to set the problem which governs the quasi-static equilibrium of the confined plasma in three dimensional domain.

Problem 5.1. Let Ω be a domain in the rectangular system $(\vec{e}_r, \vec{e}_z)$, closed domain in a boundary band $0 < r_1 \le r \le r_2 < +\infty$, $\Gamma_p = \partial\Omega_p$ is a curve sufficiently smooth. Determine $u \in C^2(\Omega) \cap C^3(\overline{\Omega})$, $\gamma_u \in R$, which satisfies the equation:

$$\mathrm{A}u(r,z) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}(r,z)\right) - \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial u}{\partial z}(r,z)\right) = \frac{1}{2r}g'(u) + \mu_0 Crf'(u),\ (\forall)(r,z) \in \Omega_p,$$

$$\mathrm{A}u(r,z) = 0, (\forall)(r,y) \in \Omega - \Omega_p,$$

and the boundary conditions: $u_{|\Gamma_p} = 0$, $\int_{\Gamma_p} \frac{1}{r}\frac{\partial u}{\partial n} ds = \mu_0 I = ct. > 0$, $u_{|\Gamma} = \gamma_u > 0$, u do not go to zero on Ω_p, $\frac{\partial u}{\partial n}$ is continuous in crossing the boundary Γ_p, where the arbitrary functions f, g satisfy the relations $f(0) = f'(0) = g'(0) = 0$.

Remark 5.1. The functions f and g are introduced with the idea of identifying the equipotential surfaces of the magnetic induction field, of the electric current field and even with the level surfaces of the hydrodynamic potential of the totally ionized gas. Indeed, we consider the equi-potential surface of the electric current $\frac{dx}{j_x} = \frac{dy}{j_y} = \frac{dz}{j_z}$, which rewritten in the Cartesian coordinate $(\vec{e}_r, \vec{e}_\varphi, \vec{e}_z)$ is $\frac{rdr}{-\frac{\partial \psi}{\partial z}} = \frac{rd\varphi}{-Au(r,z)} = \frac{rdz}{\frac{\partial \psi}{\partial r}}$. From the first and the last fraction we deduce $r\left(\frac{\partial \psi}{\partial r}dr + \frac{\partial \psi}{\partial z}dz\right) = 0$, that is $d\psi(r,z) = 0$, so $\psi(r,z) = ct.$ on the surface on which $\vec{j}$ is tangent field. On such surfaces we have $j_r = j_z = 0$, $j_\varphi \ne 0$.

On the level surface $u = ct.$ it also results $\psi = ct.$, so that the surfaces on which $\vec{B}$ and $\vec{j}$ are tangent, are part of the same family and the functions $u, \frac{\partial u}{\partial n}$ are continuous when crossing any level surface. Alone, the component j_φ is nonempty on the level surfaces from Ω_p, it is the only component which contributes to the total current in the jet, current which is observed at the free boundary through $I\mu_0 = \int_{\Gamma_p} \frac{1}{r}\frac{\partial u}{\partial n} ds$. The quadratic functions case f, g, it was study by *Temam* [17].

Following, there will be showed the relaxing ways, or generalizing the results regarding the nonlinearity of functions f, g. We will refer to the general treating from *C. Ghiţă* [4], [5].

For the particular case showed above, in order to realize a unitary and coherent treatment, we have introduced the state function of the system $h : R_{-} \times R^2 \to R$, $h(r, z; u(r, z)) = \begin{cases} \frac{1}{2r} g'(u(r, z)) + rf'(u(r, z)), (r, z) \in \Omega_p \\ 0, (r, z) \in \Omega - \Omega_p \end{cases}$ and the controlled parameter $\lambda = C\mu_0 \in R_{+}$, in agreement to which the equilibrium equation of the plasma jet is written $\mathrm{A}u(r, z) = \lambda h(r, z; u(r, z))$, as a nonlinear eigenvalue equation for the operator A.

The result of the last Theorem, demonstrated in *Temam* [18], for n=2, is extended to $n > 2$, $\beta = \dfrac{2n}{n-2}$, less the analytical function character for *u*, example [8], n = 2. Essential arguments which lead to the result are there related to the ellipticity property of operator *L*, the properties of the asymptotic behavior of the state operator B. The regularity assured by the emerge theorems allowed us to obtain the identification of the weak solution with the classic solution, and also a good regularity for the free boundary Γ_p.

The character of the eigenvalue problem was eliminated by equivalence with the problem of optimum on level surface determined by the state function of the ionized gas jet. In *Temam* [17] it is obtained the result even through a optimum problem without restriction, introducing a nonlinear functional

$$J : W \to R, J(v) = \frac{1}{2} a(v, v) - \gamma_v I + \int_\Omega F(x; \vartheta(v)) dx, \text{ with } \vartheta(v) = -v_{-}.$$

For a similar problem, using variational formulations in duality, the topological methods and a fixed point argument were obtained existence results in [2].

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