SOME INEQUALITIES IN THE CONVEX QUADRILATERAL

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Abstract: *In this paper we present some inequalities about the convex quadrilateral using the Jensen Inequality, the Tóth-Lenhard Inequality.*

Keywords: *convex quadrilateral, Jensen Inequality, Tóth-Lenhard Inequality*

1. INTRODUCTION

Let $A_1A_2A_3A_4$ be a convex quadrilateral and *M* an interior point.

Florian [1-3] proved the following inequality for a convex quadrilateral:

$$\sqrt{2}(w_1 + w_2 + w_3 + w_4) \le MA_1 + MA_2 + MA_3 + MA_4 , \tag{1}$$

where w_1, w_2, w_3 and w_4 are the bisectors of the angles $A_1\hat{M}A_2$, $A_2\hat{M}A_3$, $A_3\hat{M}A_4$ and $A_4\hat{M}A_1$ respectively.

But $p_1 \le w_1$, $p_2 \le w_2$, $p_3 \le w_3$, $p_4 \le w_4$, where p_1, p_2, p_3, p_4, are the distances from *M* to the sides of the convex quadrilateral $A_1A_2A_3A_4$ (p_1 is the distance from *M* to A_1A_2 etc), hence:

$$\sqrt{2}(p_1 + p_2 + p_3 + p_4) \le MA_1 + MA_2 + MA_3 + MA_4 . \tag{2}$$

Other inequality of Erdös-Mordell-type, for the convex quadrilateral, was given by N. Ozeki [4] in 1957, namely,

$$4 \cdot w_1 \cdot w_2 \cdot w_3 \cdot w_4 \le MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 , \tag{3}$$

which proved the inequality: $4 \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \le MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 .$ (4)

Denote by R_1, R_2, R_3, R_4 the radii of circles (circumscribing the triangles) MA_1A_2, MA_2A_3, MA_3A_4, MA_4A_1 respectively. Next, we establish some inequalities similar of about inequalities, between the lengths MA_1, MA_2, MA_3, MA_4 and the radii R_1, R_2, R_3, R_4 .

2. MAIN RESULTS

Theorem 2.1. *For every convex quadrilateral* $A_1A_2A_3A_4$ *the following inequality*

$$MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 \le 4R_1R_2R_3R_4 , \tag{5}$$

holds.

Proof. By applying the sine rule, we deduce the relations (see figure 1)

$$MA_2 = 2R_1 \sin A_{1,1} \text{ şi}$$

$$MA_2 = 2R_2 \sin A_{3,2} ,$$

hence: $\dfrac{MA_2^2}{4R_1R_2} = \sin A_{1,1} \sin A_{3,2}$

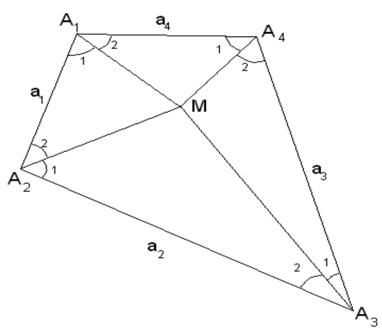

Fig. 1.

Similarly, we obtain the equalities:

$\frac{MA_3^2}{4R_2R_3}=\sin A_{2,1}\sin A_{4,2}$, $\frac{MA_4^2}{4R_3R_4}=\sin A_{3,1}\sin A_{1,2}$ and $\frac{MA_1^2}{4R_4R_1}=\sin A_{4,1}\sin A_{2,2}$.

Taking the product of the four equalities, we obtain

$$\left(\frac{MA_1\cdot MA_2\cdot MA_3\cdot MA_4}{16R_1R_2R_3R_4}\right)^2=\sin A_{1,1}\sin A_{1,2}\sin A_{2,1}\sin A_{2,2}\sin A_{3,1}\sin A_{3,2}\sin A_{4,1}\sin A_{4,2},$$

But when applying the Jensen inequality for the concave function, *ln (sinx)*, it follows that

$$\sin A_{1,1}\sin A_{1,2}\sin A_{2,1}\sin A_{2,2}\sin A_{3,1}\sin A_{3,2}\sin A_{4,1}\sin A_{4,2}\le\frac{1}{16},$$

so: $\frac{MA_1\cdot MA_2\cdot MA_3\cdot MA_4}{16R_1R_2R_3R_4}\le\frac{1}{4}$,

consequently: $MA_1\cdot MA_2\cdot MA_3\cdot MA_4\le 4R_1R_2R_3R_4$.

Theorem 2.2. *In any convex quadrilateral $A_1A_2A_3A_4$ there is the inequality*

$$MA_1\cdot MA_2\cdot MA_3\cdot MA_4\le 16R_1R_2R_3R_4\sin\frac{A_1}{2}\sin\frac{A_2}{2}\sin\frac{A_3}{2}\sin\frac{A_4}{2}. \qquad (6)$$

Proof. We have the relation

$$\frac{MA_2}{2R_1}+\frac{MA_4}{2R_4}=\sin A_{1,1}+\sin A_{1,2}\le 2\sin\frac{A_{1,1}+A_{1,2}}{2}=2\sin\frac{A_1}{2}, \qquad (7)$$

but:

$$\sqrt{\frac{MA_2\cdot MA_4}{R_1\cdot R_4}}\le 2\sin\frac{A_1}{2}, \qquad (8)$$

which means that: $\frac{MA_1\cdot MA_2\cdot MA_3\cdot MA_4}{16R_1R_2R_3R_4}\le\sin\frac{A_1}{2}\sin\frac{A_2}{2}\sin\frac{A_3}{2}\sin\frac{A_4}{2}$,

so: $MA_1\cdot MA_2\cdot MA_3\cdot MA_4\le 16R_1R_2R_3R_4\sin\frac{A_1}{2}\sin\frac{A_2}{2}\sin\frac{A_3}{2}\sin\frac{A_4}{2}$.

Theorem 2.3. *If $x_1,x_2,x_3,x_4\in R_+^*$, then in any convex quadrilateral $A_1A_2A_3A_4$ there are the inequalities*

$$\left(\frac{x_1x_2}{\sqrt{R_1R_4}}+\frac{x_3x_4}{\sqrt{R_2R_3}}\right)\sqrt{MA_2\cdot MA_4}+\left(\frac{x_2x_3}{\sqrt{R_2R_1}}+\frac{x_4x_1}{\sqrt{R_3R_4}}\right)\sqrt{MA_3\cdot MA_1}\le$$

$$\le\frac{1}{2}[\left(\frac{x_2x_3}{R_1}+\frac{x_4x_1}{R_4}\right)MA_1+\left(\frac{x_1x_2}{R_1}+\frac{x_3x_4}{R_2}\right)MA_2+\left(\frac{x_2x_3}{R_2}+\frac{x_4x_1}{R_3}\right)MA_3+$$

$$+\left(\frac{x_1x_2}{R_4}+\frac{x_3x_4}{R_3}\right)MA_4]\le\sqrt{2}\left(x_1^2+x_2^2+x_3^2+x_4^2\right). \qquad (9)$$

Proof. We know from [1-3] the Tóth- Lenhard Inequality:

Let $x_1,x_2,...,x_n$ and $\theta_1,\theta_2,...,\theta_n$ be two sets of positive real numbers so that

$$\theta_1+\theta_2+...+\theta_n=\pi.$$

If we put $x_{n+1}=x_1$, then: $\sum_{k=1}^{n}x_kx_{k+1}\cos\theta_k\le\cos\left(\frac{\pi}{n}\right)\sum_{k=1}^{n}x_k^2$.

We choose $n=4$ and $\theta_k=\frac{\pi-A_k}{2}$, $(\forall)k=\overline{1,4}$ .We remark that $\theta_1+\theta_2+\theta_3+\theta_4=\pi$.

Hence we applying the Tóth- Lenhard Inequality, we have:

$$\sum_{k=1}^{4} x_k x_{k+1} \sin\frac{A_k}{2} = \sum_{k=1}^{4} x_k x_{k+1} \cos\theta_k \le \cos\left(\frac{\pi}{4}\right)\sum_{k=1}^{4} x_k^2 ,$$

so:

$$\sum_{k=1}^{4} x_k x_{k+1} \sin\frac{A_k}{2} \le \frac{\sqrt{2}}{2}\sum_{k=1}^{4} x_k^2 . \tag{10}$$

We multiply the relations (7) and (8) with $x_1 x_2$, hence

$$x_1 x_2 \sqrt{\frac{MA_2 \cdot MA_4}{R_1 \cdot R_4}} \le \left(\frac{MA_2}{2R_1} + \frac{MA_4}{2R_4}\right) x_1 x_2 \le 2x_1 x_2 \sin\frac{A_1}{2} .$$

We written the similarly relation and taking the sum, we obtain the sequence of inequalities

$$\left(\frac{x_1x_2}{\sqrt{R_1R_4}} + \frac{x_3x_4}{\sqrt{R_2R_3}}\right)\sqrt{MA_2 \cdot MA_4} + \left(\frac{x_2x_3}{\sqrt{R_2R_1}} + \frac{x_4x_1}{\sqrt{R_3R_4}}\right)\sqrt{MA_3 \cdot MA_1} \le$$

$$\le \frac{1}{2}[\left(\frac{x_2x_3}{R_1} + \frac{x_4x_1}{R_4}\right)MA_1 + \left(\frac{x_1x_2}{R_1} + \frac{x_3x_4}{R_2}\right)MA_2 + \left(\frac{x_2x_3}{R_2} + \frac{x_4x_1}{R_3}\right)MA_3 +$$

$$+\left(\frac{x_1x_2}{R_4} + \frac{x_3x_4}{R_3}\right)MA_4] \le$$

$$\le 2x_1x_2 \sin\frac{A_1}{2} + 2x_2x_3 \sin\frac{A_2}{2} + 2x_3x_4 \sin\frac{A_2}{2} + 2x_4x_1 \sin\frac{A_2}{2} \le \sqrt{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right).$$

Therefore, we obtain the inequalities of statement.

Corollary 2.4. *In any convex quadrilateral* $A_1A_2A_3A_4$*, the inequalities*

$$\left(\frac{1}{\sqrt{R_1R_4}} + \frac{1}{\sqrt{R_2R_3}}\right)\sqrt{MA_2 \cdot MA_4} + \left(\frac{1}{\sqrt{R_2R_1}} + \frac{1}{\sqrt{R_3R_4}}\right)\sqrt{MA_3 \cdot MA_1} \le$$

$$\le \frac{1}{2}[\left(\frac{1}{R_1} + \frac{1}{R_4}\right)MA_1 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)MA_2 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)MA_3 + \left(\frac{1}{R_4} + \frac{1}{R_3}\right)MA_4] \le 4\sqrt{2} \tag{11}$$

and

$$\left(\sqrt{\frac{R_2}{R_4}} + \sqrt{\frac{R_4}{R_2}}\right)\sqrt{MA_2 \cdot MA_4} + \left(\sqrt{\frac{R_1}{R_3}} + \sqrt{\frac{R_3}{R_1}}\right)\sqrt{MA_1 \cdot MA_3} \le$$

$$\le \frac{1}{2}[\left(\frac{\sqrt{R_2R_3}}{R_1} + \sqrt{\frac{R_1}{R_4}}\right)MA_1 + \left(\frac{\sqrt{R_3R_4}}{R_2} + \sqrt{\frac{R_2}{R_1}}\right)MA_2 + \left(\frac{\sqrt{R_4R_1}}{R_3} + \sqrt{\frac{R_3}{R_2}}\right)MA_3 +$$

$$+\left(\frac{\sqrt{R_1R_2}}{R_4} + \sqrt{\frac{R_4}{R_3}}\right)MA_4] \le \sqrt{2}\left(R_1 + R_2 + R_3 + R_4\right) \tag{12}$$

hold.

Proof. It follows from inequality (9) if we put $x_1 = x_2 = x_3 = x_4 = 1$ and $x_1 = \sqrt{R_1}$, $x_2 = \sqrt{R_2}$, $x_3 = \sqrt{R_3}$, $x_4 = \sqrt{R_4}$ respectively.

Corollary 2.5. *in any convex quadrilateral* $A_1A_2A_3A_4$ *there are the inequalities*

$$\frac{MA_1}{\sqrt{R_4R_1}} + \frac{MA_2}{\sqrt{R_1R_2}} + \frac{MA_3}{\sqrt{R_2R_3}} + \frac{MA_4}{\sqrt{R_3R_4}} \le 4\sqrt{2} \tag{13}$$

and $$\sqrt{2}\left(\sqrt{MA_1 \cdot MA_3} + \sqrt{MA_2 \cdot MA_4}\right) \le R_1 + R_2 + R_3 + R_4 \ . \qquad (14)$$

Proof. From the relation (11), we have

$$\left(\frac{1}{R_1} + \frac{1}{R_4}\right)MA_1 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)MA_2 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)MA_3 + \left(\frac{1}{R_4} + \frac{1}{R_3}\right)MA_4 \le 8\sqrt{2}$$

and the relation $\frac{1}{u} + \frac{1}{v} \ge \frac{2}{\sqrt{uv}}$, for all $u, v > 0$, implies that

$$\frac{MA_1}{\sqrt{R_4 R_1}} + \frac{MA_2}{\sqrt{R_1 R_2}} + \frac{MA_3}{\sqrt{R_2 R_3}} + \frac{MA_4}{\sqrt{R_3 R_4}} \le 4\sqrt{2} \ .$$

From the relation (11), we have

$$\left(\sqrt{\frac{R_2}{R_4}} + \sqrt{\frac{R_4}{R_2}}\right)\sqrt{MA_2 \cdot MA_4} + \left(\sqrt{\frac{R_1}{R_3}} + \sqrt{\frac{R_3}{R_1}}\right)\sqrt{MA_1 \cdot MA_3} \le \sqrt{2}\left(R_1 + R_2 + R_3 + R_4\right).$$

But, using the inequality $\sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}} \ge 2$, for all $u, v > 0$, we deduce inequality (14).

Remark. *From inequality (13), we deduce another inequality*

$$\frac{MA_1}{R_1 + R_4} + \frac{MA_2}{R_1 + R_2} + \frac{MA_3}{R_2 + R_3} + \frac{MA_4}{R_3 + R_4} \le 2\sqrt{2} \ . \qquad (15)$$

Corollary 2.6. *In any convex quadrilateral $A_1 A_2 A_3 A_4$, the inequalities*

$$\sqrt{2}\left(\sqrt{MA_1 \cdot MA_3} + \sqrt{MA_2 \cdot MA_4}\right) \le \sqrt{R_1 R_2 R_3 R_4}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) \qquad (16)$$

and

$$\left(R_2 + R_3\right)MA_1 + \left(R_3 + R_4\right)MA_2 + \left(R_4 + R_1\right)MA_3 + \left(R_1 + R_2\right)MA_4 \le \ \le 2\sqrt{2}R_1 R_2 R_3 R_4\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} + \frac{1}{R_4^2}\right)$$

hold.

Proof. It easy to see that from inequality (9) that for $x_1 = \frac{1}{\sqrt{R_2}}$, $x_2 = \frac{1}{\sqrt{R_3}}$, $x_3 = \frac{1}{\sqrt{R_4}}$, $x_4 = \frac{1}{\sqrt{R_1}}$ and $x_1 = \frac{1}{R_2}$, $x_2 = \frac{1}{R_3}$, $x_3 = \frac{1}{R_4}$, $x_4 = \frac{1}{R_1}$ respectively, we obtain inequalities (16) and (17).

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