ON THE IMAGES OF SOME OPEN SETS HAVING THE NON-EMPTY INTERSECTION PROPERTY

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Abstract. *Let E and F be normed spaces, T:E→F and V:E→F satisfying* $T(O_1)\bigcap V(O_2)\neq\phi$ *for all* O_1, O_2 *non-emty open subsets of E. In this paper we present some aspects about the properties of the functions T and V.*

Keywords*: normed space, discontinuous function, boundness.*

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1. INTRODUCTION

In the paper [1] it is considered functions $f,g:\mathbb{R}\to\mathbb{R}$, which satisfy a condition of the type $f(I)\cap g(J)\neq\phi$ where $I,J\subseteq\mathbb{R}$ are non-empty open intervals. It is studied some properties of the functions f and g about their continuity.

In the monograph [2] it is considered functions $u,v:X\to X$ (where X is a metric space) satisfying $u(K_1)\cap v(K_2)\neq\phi$ for all K_1,K_2 non-empty compact subsets of X and some properties about their boundness are given.

Let E and F be normed spaces endowed with the norms $\|\cdot\|_E$, respectively $\|\cdot\|_F$. We consider two functions $T:E\to F$, $V:E\to F$ satisfying the following property:

$$T(O_1)\cap T(O_2)\neq\phi \text{ for all } O_1,\ O_2 \text{ non-empty subsets of } E \tag{1}$$

In this paper, firstly, we study the continuity of T, when V satisfies an unboundness condition, extending one of the results from [1]. Secondly we give some remarks about the result obtained in the first part of this paper.

The monograph [3] appears to the References only for nominating one of the many books in which there are presented the basic facts regarding normed spaces, continuity and boundness.

2. RESULTS

Theorem 2.1 *Let* $T:E\to F$, $V:E\to F$ *satisfying* (1). *If* $\lim_{\|x\|_E\to\infty}\|V(x)\|_F=\infty$, *then* T *is discontinuous at every point* $a\in E$.

Proof: Let $a\in E$ and n be a natural number, $n\geq 1$.

Let $A_n=B_E(a,\frac{1}{n})=\left\{x\in E/\|x-a\|_E<\frac{1}{n}\right\}$ and $B_n=E-\bar{B}(0,n)=E-\left\{x\in E/\|x\|_E\leq n\right\}$.

A_n and B_n are non-empty open subsets of E, and, using (1), we obtain that

$T(A_n)\cap V(B_n)=T(B_E(a,\frac{1}{n}))\cap V(E-\bar{B}_E(0,n))\neq\phi$.

It results that it exists $y_n\in T(A_n)\cap V(B_n)$. Consequently $y_n=T(u_n)=V(v_n)$, where $u_n\in B_E(a,\frac{1}{n})$ and $v_n\in E-\bar{B}(0,n)$.

So we have obtained two sequences $(u_n)_n$, $(v_n)_n \subset E$ with the properties $u_n \to a$ and $\|v_n\|_e \to \infty$.

From $\lim\limits_{\|x\|_E \to \infty} \|V(x)\|_F = \infty$, we obtain that $\|y_n\|_F = \|V(v_n)\|_F \to \infty$.

If T is continuous at the point a, then we have $y_n = T(u_n) \to T(a)$, i.e. $\|y_n\|_F \to \|T(a)\|_F$ and this is contradictory to the fact that $\|y_n\|_F \to \infty$. Thus the proof of the *Theorem 2.1* is complete.

3. REMARKS

The property (1) being satisfied, we are interested in studying the boundness of V ,in the sense that $\operatorname{Im} V = \{V(x) / x \in E\}$ is bounded, if T has at least one point of continuity.

The answer is that V is not necessarily bounded, in the specified sense, which can be observed from the following example:

If T and V are functions from the real numbers set into itself, $T(x) = 0$ for all rationals, $T(x) = x^2$ for all irrationals, $V(x) = 0$ for all rationals and $V(x) = x$ for all irrationals, then the property (1) is satisfied, T is continuous only at the point $a = 0$ and V is not bounded, in the specified sense.

However, we can obtain the following boundness result for V :

Theorem 3.1 *Let* $T : E \to F$, $V : E \to F$ *satisfying* (1)*. If* T *is continuous at the point* $a \in E$ *and* $(u_n)_n \subset E$ *with* $\|u_n\|_E \to \infty$*, then it exists the sequence* $(t_n)_n \subset E$ *so that* $\|t_n - u_n\|_E \to 0$ *and the sequence* $(V(t_n))_n$ *is bounded.*

Proof: Let $(u_n)_n \subset E$ with $\|u_n\|_E \to \infty$. Let n be a natural number, $n \geq 1$. From (1) we obtain $T(B_E(a, \frac{1}{n})) \cap V(B_E(u_n, \frac{1}{n})) \neq \phi$.

It results that it exists $y_n \in T(B_E(a, \frac{1}{n})) \cap V(B_E(u_n, \frac{1}{n}))$.

Consequently $y_n = T(x_n) = V(t_n)$, where $x_n \in B_E(a, \frac{1}{n})$ and $t_n \in B_E(u_n, \frac{1}{n})$. So we have obtained two sequences $(x_n)_n, (t_n)_n \subset E$ with the properties $x_n \to a$ and $\|t_n - u_n\|_E \to 0$. From the continuity of T at the point a, we obtain that $y_n = T(x_n) \to T(a)$, and consequently, the sequence $(V(t_n))_n$ is bounded. Thus our assertion is proved.

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