A GENERALIZATION OF BERGSTROM AND RADON'S INEQUALITIES IN PSEUDO-HILBERT SPACES

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Abstract. *In this note we had presented two generalizations for Bergstrom and Radon's inequalities for seminorms in pseudo-Hilbert spaces and in normed spaces. Some applications are given, as well.*

Keywords: *pseudo-Hilbert spaces (Loynes spaces), seminorms, Bergstrom inequality, Radon inequality.*

INTRODUCTION

First we need to recall, see [3, 5], that a locally convex space **Z** is called admissible in the Loynes sense if the following conditions are satisfied:

- **Z** is complete;
- there is a closed convex cone in **Z** , denoted $\mathbf{Z}_+$, that defines an order relation on **Z** (that is $z_1 \le z_2$ if $z_1 - z_2 \in \mathbf{Z}_+$);
- there is an involution in **Z**, $\mathbf{Z} \ni z \to z^* \in \mathbf{Z}$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in \mathbf{Z}_+$ implies $z^* = z$;
- the topology of **Z** is compatible with the order (that is, there exists a basis of convex solid neighbourhoods of the origin);
- and any monotonously decreasing sequence in $\mathbf{Z}_+$ is convergent.

We shall say that a set $C \in \mathbf{Z}$ is called solid if $0 \le z' \le z''$ and $z'' \in C$ implies $z' \in C$.

As an easy example we shall consider, $\mathbf{Z} = C$, a C^* - algebra with topology and natural involution.

Let **Z** be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes **Z** – space if it satisfies the following properties:

- $\mathcal{H}$ is endowed with a **Z** - valued inner product (gramian), i.e. there exists an application

$\mathcal{H} \times \mathcal{H} \ni (h, k) \to [h, k] \in \mathbf{Z}$ having the properties: $[h, h] \geq 0$; $[h, h]=0$ implies $h=0$;
$[h_1+h_2, h]=[h_1, h]+[h_2 ,h]$; $[\lambda h, k]= \lambda [h, k]$; $[h, k]^* = [k, h]$;
for all $h, k, h_1 ,h_2 \in H$ and $\lambda \in \mathbf{C}$.

- The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h, h] \in \mathbf{Z}$ is continuous. Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes **Z**-- *space*.

Now, considering **Z**=C as above, **Z** with $[z_1, z_2] = z_2^* z_1$ is a Loynes **Z** -space.

An important result which can be used below is given in the next statement, and was proved in [5].

Let $\mathcal{H}$ and $\mathcal{K}$ be two Loynes **Z**-spaces.

We recall that in [3-5] an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called gramian bounded, if there exists a constant $\mu>0$ such that in the sense of order of **Z** holds

$$[Th, Th]_{\mathbf{K}} \le \mu\, [h, h]_{\mathbf{H}}, \; h \in \mathcal{H}. \tag{1}$$

We denote the class of such operators by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and
$\mathcal{B}^*(\mathcal{H}, \mathcal{K}) = \mathcal{B}(\mathcal{H}, \mathcal{K}) \cap \mathcal{L}^*(\mathcal{H}, \mathcal{K})$.

We also denote the introduced norm by

$\|T\| = \inf\{\sqrt{\mu},\ \mu>0 \text{ and satisfies } (1)\}$. (2)

It is known that the space $\mathcal{B}^*(\mathcal{H}, \mathcal{K})$ is a Banach space, and its involution $\mathcal{B}^*(\mathcal{H}, \mathcal{K})$ in $\mathcal{B}^*(\mathcal{K}, \mathcal{H})$ satisfies

$\|T^*T\| = \|T\|^2$, $T \in \mathcal{B}^*(\mathcal{H}, \mathcal{K})$.

In particular $\mathcal{B}^*(\mathcal{H})$ is a C^*--algebra.

The following two results were presented in [3].

Lemma 1. If p is a continuous and monotonous seminorm on **Z**, then:
$q_p(h) = (p([h,h]))^{\frac{1}{2}}$ is a continuous seminorm on $\mathcal{H}$.

Proposition 1. If $\mathcal{H}$ is a pre-Loynes **Z**-space and $\mathcal{P}$ is a set of monotonous (increasing) seminorms defining the topology of **Z**, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathbf{P}} = \{q_p \mid p \in \mathcal{P}\}$.

We suppose that $mq_{p_2}(x) \le q_{p_1}(x) \le Mq_{p_2}(x)$, $(\forall)$ $x \in \mathcal{H}$, with p_1, p_2 continuous and increasing seminorms on **Z** and M finite, $M \ge m > 0$. Because p_2 is increasing, we have:

$$\frac{M^2}{m^2}\left\{2+\frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\right\} \ge q_{p_2}^2\left(\frac{x}{q_{p_2}(x)}+\frac{y}{q_{p_2}(y)}\right) \ge 2\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)}$$

Thus, for example,

$$\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)} \le \frac{M^2}{m^2}\left\{1+\frac{1}{2}\frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\right\}.$$

Let **Z** be an admissible space in the Loynes sense and $\mathcal{H}$ is a pre-Loynes **Z**-space. Using the Radon's inequality we can state:

Remark 1. If $h_k \in \mathcal{H}$, $a_k>0$ with $q_p(h_k)>0$, $r>0$, $k \in \{1,2,...,n\}$ then we shall have:

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+1}}{a_k^r} \ge \frac{\left(\sum_{k=1}^{n} q_p(h_k)\right)^{r+1}}{\left(\sum_{k=1}^{n} a_k\right)^r}$$

When we take r = 1 a variant of Bergstrom's inequality with seminorm is obtained.

2. THE MAIN RESULTS

The proofs of the Theorems 1 and 4 will use the same techniques as in [6].

Theorem 1. For $a_k, h_k \in \mathcal{H}$ with $q_p(h_k) > 0$, $r \ge 1$, $k \in \{1, 2, ...,n\}$, $n \ge 2$, $n \in \mathbf{N}$ the following inequality takes place:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} q_p(h_k))^{r+1}}{\left(\sum_{k=1}^{n} a_k\right)^r} + \max_{\mathbf{1\leq i<j\leq n}}\left(\frac{q_p(h_i)^{r+1}}{a_i^r} + \frac{q_p(h_j)^{r+1}}{a_j^r} - \frac{q_p(h_i+h_j)^{r+1}}{(a_i+a_j)^r}\right)$$

Proof: We shall consider the sequence,

$$d_n = \frac{q_p(h_1)^{r+1}}{a_1^r} + \ldots + \frac{q_p(h_n)^{r+1}}{a_n^r} - \frac{q_p(h_1+\ldots+h_n)^{r+1}}{(a_1+\ldots+a_n)^r}$$

and we shall prove that $(d_n)_n$ is an increasing monotonous sequence.

This fact it indeed true if consider

$$d_{n+1} - d_n = \frac{q_p(h_{n+1})^{r+1}}{a_{n+1}^r} - \frac{q_p(h_1+\ldots+h_n+h_{n+1})^{r+1}}{(a_1+\ldots+a_n+a_{n+1})^r} + \frac{q_p(h_1+\ldots+h_n)^{r+1}}{(a_1+\ldots+a_n)^r}$$

because q_p is seminorm which implies,

$$q_p(h_1+\ldots+h_n+h_{n+1}) \leq q_p(h_1+\ldots+h_n) + q_p(h_{n+1})$$

and also:

$$\frac{(q_p(h_1+\ldots+h_n+h_{n+1}))^{r+1}}{((a_1+\ldots+a_n)+a_{n+1})^r} \leq \frac{(q_p(h_1+\ldots+h_n)+q_p(h_{n+1}))^{r+1}}{((a_1+\ldots+a_n)+a_{n+1})^r} \leq \frac{q_p(h_1+\ldots+h_n)^{r+1}}{(a_1+\ldots+a_n)^r} + \frac{q_p(h_{n+1})^{r+1}}{a_{n+1}^r}.$$

We used before the Radon's inequality applied for n=2, see [1, 6],

$$\frac{\alpha^{r+1}}{a^r} + \frac{\beta^{r+1}}{b^r} \geq \frac{(\alpha+\beta)^{r+1}}{(a+b)^r}, \tag{3}$$

and we took $\alpha = q_p(h_1+\ldots+h_n)$, $\beta = q_p(h_{n+1})$, $a = a_1 +\ldots+ a_n$ and $b = a_{n+1}$

Another proof for inequality (3), can be found in [1].

The sequence $(d_n)_n$ being increasing, we obtain that,

$$d_n \geq d_{n-1} \geq \ldots \geq d_2 \geq d_1 = 0$$

and that also means that

$$d_n \geq d_2 = \frac{q_p(h_1)^{r+1}}{a_1^r} + \frac{q_p(h_2)^{r+1}}{a_2^r} - \frac{q_p(h_1+h_2)^{r+1}}{(a_1+a_2)^r}, \quad (\forall) n \in \mathrm{N}, \quad n \geq 2.$$

The symmetry of d_n relatively to the variables a_i and h_j, $i, j \in \{1, 2,..., n\}$ allows us to notice that:

$$d_n \geq \frac{q_p(h_i)^{r+1}}{a_i^r} + \frac{q_p(h_j)^{r+1}}{a_j^r} - \frac{q_p(h_i+h_j)^{r+1}}{(a_i+a_j)^r}, \quad (\forall) n \in \mathrm{N}, \quad n \geq 2, i, j \in \{1,2,\ldots,n\}.$$

For r=1 is obtained below a refinement of Bergstrom's inequality.

Corollary 1. For $a_k > 0$, $h_k \in \mathcal{H}$, $k \in \{1, 2, ..., n\}$, $n \geq 2$, $n \in \mathrm{N}$ we shall obtain the following inequality:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^2}{a_k}\geq\frac{(\sum_{k=1}^{n}q_p(h_k))^2}{(\sum_{k=1}^{n}a_k)}+\max_{1\leq i<j\leq n}\frac{(a_i+a_j)(a_jq_p^2(h_i)+a_iq_p^2(h_j))-a_ia_jq_p^2(h_i+h_j)}{a_ia_j(a_i+a_j)}$$

Theorem 2. For $a_k > 0$, $x_k \in \boldsymbol{X}$, $r \geq 0$, $k\in\{1, 2, ..., n\}$, $n\geq 2$, $n\in N$ and every arbitrary seminorm p, $p:\boldsymbol{X}\rightarrow R_+$ we have:

$$\sum_{k=1}^{n}\frac{p(x_k)^{r+1}}{a_k^r}\geq\frac{(\sum_{k=1}^{n}p(x_k))^{r+1}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{p(x_i)^{r+1}}{a_i^r}+\frac{p(x_j)^{r+1}}{a_j^r}-\frac{p(x_i+x_j)^{r+1}}{(a_i+a_j)^r}\right).$$

Corollary 2. In fact with the above conditions, the Corollary 1 remains true for every seminorm p from a family of seminorms which defines the topology of the linear space considered instead of q_p.

Theorem 3. If we consider a normed space $\mathcal{H}$, $x_k\in\mathcal{H}$, $k\in\{1, 2,..., n\}$ and with the above conditions of the Theorem 1, then we have the following inequality:

$$\sum_{k=1}^{n}\frac{\|x_k\|^{r+1}}{a_k^r}\geq\frac{(\sum_{k=1}^{n}\|x_k\|)^{r+1}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{\|x_i\|^{r+1}}{a_i^r}+\frac{\|x_j\|^{r+1}}{a_j^r}-\frac{\|x_i+x_j\|^{r+1}}{(a_i+a_j)^r}\right).$$

Proof:

It will be as the proof of the Theorem 1, we shall only take

$$d_n=\frac{\|x_1\|^{r+1}}{a_1^r}+\ldots+\frac{\|x_n\|^{r+1}}{a_n^r}-\frac{\|x_1+\ldots+x_n\|^{r+1}}{(a_1+\ldots+a_n)^r}$$

and $\alpha=\|x_1+\ldots+x_n\|$, $\beta=\|x_{n+1}\|$, $a=a_1+\ldots+a_n$ and $b=a_{n+1}$ in relation (3).

Remark 2. We can consider instead of seminorm p, a norm $\|.\|$, in a normed space $\mathcal{H}$, $x_i\in\mathcal{H}$ and then under conditions of the above corollary we shall have,

$$\sum_{k=1}^{n}\frac{\|x_k\|^2}{a_k}\geq\frac{(\sum_{k=1}^{n}\|x_k\|)^2}{\sum_{k=1}^{n}a_k}+\max_{1\leq i<j\leq n}\frac{(a_i+a_j)(a_j\|x_i\|^2+a_i\|x_j\|^2)-a_ia_j\|x_i+x_j\|^2)}{a_ia_j(a_i+a_j)}.$$

In what follows we shall present a generalizations of the Remark 1 concerning the Radon's inequality for seminorms q_p.

Remark 3. If $h_k\in\mathcal{H}$, $a_k > 0$, $r > 0$, $s\geq 1$, $k\in\{1, 2, ..., n\}$, $m\geq 1$, then the following inequalities take place:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^{r+s}}{a_k^r}\geq\frac{1}{n^{s-1}}\frac{(\sum_{k=1}^{n}q_p(h_k))^{r+s}}{(\sum_{k=1}^{n}a_k)^r},\quad \sum_{k=1}^{n}q_p(h_k)^m\geq\frac{1}{n^{m-1}}\left(\sum_{k=1}^{n}q_p(h_k)\right)^m.$$

Now we shall be able to give a generalization of Theorem 1, Radon's inequality for seminorms q_p.

Theorem 4. For a_k, $h_k\in\mathcal{H}$ with $q_p(h_k)> 0$, $r \geq 0$, $s\geq 1$, $k\in\{1, 2, …, n\}$, $n\geq 2$, $n\in N$ the following inequality takes place:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^{r+s}}{a_k^r}\geq\frac{1}{n^{s-1}}\frac{(\sum_{k=1}^{n}q_p(h_k))^{r+s}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{q_p(h_i)^{r+s}}{a_i^r}+\frac{q_p(h_j)^{r+s}}{a_j^r}-\frac{q_p(h_i+h_j)^{r+s}}{(a_i+a_j)^r}\right). \tag{4}$$

Proof: We shall write:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^{r+s}}{a_k^r}\geq\sum_{k=1}^{n}\frac{(q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{a_k^r}.$$

and then applying the inequality from Theorem 1, we shall obtain,

$$\sum_{k=1}^{n}\frac{(q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{a_k^r}\geq\frac{\left(\sum_{k=1}^{n}q_p(h_k)^{\frac{r+s}{r+1}}\right)^{r+1}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{(q_p(h_i)^{\frac{r+s}{r+1}})^{r+1}}{a_i^r}+\frac{(q_p(h_j)^{\frac{r+s}{r+1}})^{r+1}}{a_j^r}-\frac{(q_p(h_i+h_j)^{\frac{r+s}{r+1}})^{r+1}}{(a_i+a_j)^r}\right).$$

This inequality becomes,

$$\sum_{k=1}^{n}\frac{(q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{a_k^r}\geq\frac{\left(\sum_{k=1}^{n}q_p(h_k)^{\frac{r+s}{r+1}}\right)^{r+1}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{q_p(h_i)^{r+s}}{a_i^r}+\frac{q_p(h_j)^{r+s}}{a_j^r}-\frac{q_p(h_i+h_j)^{r+s}}{(a_i+a_j)^r}\right).$$

Now, using the inequality from Remark 3, we shall obtain:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^{r+s}}{a_k^r}\geq\frac{\frac{1}{n^{s-1}}(\sum_{k=1}^{n}q_p(h_k))^{r+s}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{q_p(h_i)^{r+s}}{a_i^r}+\frac{q_p(h_j)^{r+s}}{a_j^r}-\frac{q_p(h_i+h_j)^{r+s}}{(a_i+a_j)^r}\right).$$

Remark 4.

- In fact under the above conditions, the above theorem remains true for every seminorm p from a family of seminorms which defines the topology of the linear space considered instead of q_p:

$$\sum_{k=1}^{n}\frac{p(x_k)^{r+s}}{a_k^r}\geq\frac{1}{n^{s-1}}\frac{(\sum_{k=1}^{n}p(x_k))^{r+s}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{p(x_i)^{r+s}}{a_i^r}+\frac{p(x_j)^{r+s}}{a_j^r}-\frac{p(x_i+x_j)^{r+s}}{(a_i+a_j)^r}\right), \tag{5}$$

$(\forall)x_k\in X$, with $p(x_k)>0$.

- Moreover, in every normed space *X*, we have under above conditions,

$$\sum_{k=1}^{n}\frac{\|x_k\|^{r+s}}{a_k^r}\geq\frac{1}{n^{s-1}}\frac{(\sum_{k=1}^{n}\|x_k\|)^{r+s}}{(\sum_{k=1}^{n}a_k)^r}+\max_{1\leq i<j\leq n}\left(\frac{\|x_i\|^{r+s}}{a_i^r}+\frac{\|x_j\|^{r+s}}{a_j^r}-\frac{\|x_i+x_j\|^{r+s}}{(a_i+a_j)^r}\right), \tag{6}$$

$(\forall)x_k\in X$.

- Finally, for a_k, $h_k\in\mathcal{H}$ with $q_p(h_k)>0$, $r\geq 0$, $s\geq r+1$, $k\in\{1, 2, ..., n\}$, $n\geq 2$, $n\in N$ a variant of Radon's inequality takes place:

$$\sum_{k=1}^{n}\frac{q_p(h_k)^s}{a_k^r} \ge \frac{1}{n^{s-r-1}}\frac{(\sum_{k=1}^{n} q_p(h_k))^s}{(\sum_{k=1}^{n} a_k)^r} + \max_{\mathbf{1 \le i < j \le n}}\left(\frac{q_p(h_i)^s}{a_i^r} + \frac{q_p(h_j)^s}{a_j^r} - \frac{q_p(h_i+h_j)^s}{(a_i+a_j)^r}\right). \tag{7}$$

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