ON THE INVERSE PROBLEM FOR AN AVERAGED OPERATOR IN HILBERT SPACES

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Abstract. *The aim of this paper is to present some properties of the averaged operators defined in Hilbert spaces. We are interested in the study of the existence of an inverse for the averaged operator associated to a strongly monotone operator.*

Keywords: *averaged operator, real Hilbert space, strongly monotone operator, non-expansive operator, firmly non-expansive operator.*

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1. INTRODUCTION

Let H be a real Hilbert space endowed with the inner product $<\cdot,\cdot>$ and the norm $\|\cdot\|$. Let $N: H \to H$ be a non-expansive operator (i.e. $\|N(x) - N(y)\| \le \|x - y\|$ for all $x, y \in H$). An operator $T: H \to H$ is called averaged if $T = (1-\alpha)I + \alpha N$ for some $\alpha \in (0,1)$, where I is the identity of H. We say that T is the averaged operator associated to the operator N.

The term "averaged operator" appears firstly in 1977 in the work [1] of Bruck and Reich. Few years later, motivated by the Krasnoselskii-Mann Theorem, the averaged operators are considered in the study of the operators having multiple fixed points. Some authors require that N is arbitrary and sometimes T is called a relaxation of the operator N.

The properties of the averaged operators have been investigated by many authors (see [2],[3],[4],[5],[6]) and, firstly, we present here the principal results and short proofs of some of them (as they appear in the book [4] of C. Byrne).

Theorem 1.1 Let $N: H \to H$ *be non-expansive,* $Fix(N) \neq \phi$ *and* $T: H \to H$ *be the averaged operator associated to* N. *Then* $Fix(T) \neq \phi$ *and* $Fix(T) = Fix(N)$.

($Fix(A)$ *denotes the set of the fixed points of the operator* A *)*

Theorem 1.2 Let $T = (1-\alpha)A + \alpha N$ *for some* $\alpha \in (0,1)$. *If* A *is averaged and* N *is non-expansive, then* T *is averaged.*

Proof: Let $A = (1-\beta)I + \beta M$ for some $\beta \in (0,1)$ and M a non-expansive operator.

Let $\gamma = 1-(1-\alpha)(1-\beta)$. Then we have

$$T = (1-\gamma)I + \gamma[(1-\alpha)\beta\gamma^{-1}M + \alpha\gamma^{-1}N].$$

Since the operator $K = (1-\alpha)\beta\gamma^{-1}M + \alpha\gamma^{-1}N$ is easily shown to be non-expansive and the convex combination of two non-expansive operators is again non-expansive, T is averaged.

Theorem 1.3 *An operator* $S: H \to H$ *is called firmly non-expansive if* $\langle S(x) - S(y), x - y \rangle \ge \|S(x) - S(y)\|^2$ *for all* $x, y \in H$. *Let* $T = (1-\alpha)F + \alpha N$ *for some* $\alpha \in (0,1)$. *If* F *is firmly non-expansive and* N *is non-expansive, then* T *is averaged.*

Theorem 1.4 *If* A *and* B *are averaged, then* $T = AB$ *is averaged.*

Theorem 1.5 *Let* A *and* B *be averaged operators and suppose that* $Fix(A) \cap Fix(B) \neq \phi$. *Then* $Fix(A) \cap Fix(B) = Fix(AB) = Fix(BA)$.

Let $V: H \to H$ be a non-expansive operator. It is known that, if V is strongly monotone(i.e. it exists $\lambda > 0$ so that $\langle V(x) - V(y), x - y \rangle \ge \lambda \|x - y\|^2$ for all $x, y \in H$; from the non-expansivity of V it is clear that, in fact, $\lambda \in (0,1)$), then V has an inverse, and consequently the equation $V(x) = f$ has an unique solution for all $f \in H$.(see [7])

In this paper we prove that the averaged operator associated to a non-expansive strongly monotone operator has an inverse.

2. RESULTS

Theorem 2.1 *Let $N : H \to H$ be a non-expansive strongly monotone operator,*

$$\|N(x) - N(y)\| \le \|x - y\| \text{ for all } x, y \in H\,;$$

$$\langle N(x) - N(y), x - y\rangle \ge \lambda \|x - y\|^2 \text{ for all } x, y \in H \ (\lambda \in (0,1)).$$

Then the averaged operator T associated to N has an inverse.

Proof: We have $T = (1-\alpha)A + \alpha N$ for some $\alpha \in (0,1)$. Consequently

$$\|T(x) - T(y)\| = \|(1-\alpha)x + \alpha N(x) - (1-\alpha)y - \alpha N(y)\|$$

$$\le (1-\alpha)\|x - y\| + \alpha \|N(x) - N(y)\| \le \|x - y\| \text{ for all } x, y \in H .$$

Also we obtain

$$\langle T(x) - T(y), x - y\rangle = \langle (1-\alpha)x + \alpha N(x) - (1-\alpha)y - \alpha N(y), x - y\rangle$$

$$= (1-\alpha)\langle x - y, x - y\rangle + \alpha \langle N(x) - N(y), x - y\rangle \ge (1 - \alpha + \alpha\lambda)\|x - y\| \text{ for all } x, y \in H .$$

$1 - \alpha + \alpha\lambda > 0$ because $\alpha \in (0,1)$ and, clearly, $1 - \alpha + \alpha\lambda < 1$. Thus T is a non-expansive strongly monotone operator. It results now that the operator T has an inverse and so, the proof of the theorem is complete.

3. REMARKS

1. The previous result shows that the averaged operator associated to an operator N preserves the non-expansivity and the strong monotonicity.
2. It's clear that some properties in this article are true for all $\alpha > 0$. Thus it is interesting to study operators of the form $(1-\alpha)A + \alpha B$ for some $\alpha > 0$, when the operators A and B have some special properties; so we can find a way to study some concrete equations.

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