

THE VORONOVSKAJA TYPE THEOREM FOR THE SZÁSZ-MIRAKJAN-KANTOROVICH OPERATORS

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Abstract. The present article continues earlier research by O. T. Pop and the author establishes a convergence theorem and the evaluation of the rate of convergence in terms of the modulus of continuity, for the well-known Szász-Mirakjan-Kantorovich operators.

Keywords: Voronovskaja's type theorem, Szász-Mirakjan operators, Kantorovich operators, Szász-Mirakjan-Kantorovich operators, modulus of continuity.

1. INTRODUCTION

Let \mathbb{N} be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this section, I recall some results from [9], which I shall use in the paper.

Let $I, J \subset \mathbb{R}$ be intervals, such that $I \cap J \neq \emptyset$. For any $n, k \in \mathbb{N}_0$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \rightarrow \mathbb{R}$, with the property $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and the linear positive functional $A_{n,k} : E(I) \rightarrow \mathbb{R}$.

For $n \in \mathbb{N}$ define the operators $L_n : E(I) \rightarrow F(J)$, by

$$(L_n f)(x) = \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(f), \quad (1)$$

where $E(I)$ and $F(J)$ are subsets of the set of real-valued functions defined on I and J , respectively.

Remark 1. The operators $(L_n)_{n \in \mathbb{N}}$ are linear and positive on $I \cap J$.

For $n \in \mathbb{N}$, $i \in \mathbb{N}_0$ define T_i by

$$(T_i L_n)(x) = n^i (L_n \psi_x^i)(x) = n^i \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(\psi_x^i), \quad x \in I \cap J. \quad (2)$$

In what follows $s \in \mathbb{N}_0$ is even and suppose that the operators $(L_n)_{n \in \mathbb{N}}$ verify the conditions:

- there exists the smallest $\alpha_s, \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{n \rightarrow \infty} \frac{(T_j L_n)(x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \quad (3)$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2, \quad (4)$$

- $I \cap J$ is an interval.

Theorem 2. ([9]) Let $f : I \rightarrow \mathbb{R}$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x , with $f^{(s)}$ continuous in x , then

$$\lim_{n \rightarrow \infty} n^{s-\alpha_s} \left((L_n f)(x) - \sum_{i=0}^s \frac{f^{(i)}(x)}{n^i \cdot i!} (T_i L_n)(x) \right) = 0. \quad (5)$$

Assume that f is s times differentiable on I , with $f^{(s)}$ continuous on I and there exists an interval $K \subset I \cap J$ such that, there exist $n(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ depending on K , so that for

$$\text{any } n \geq n(s) \text{ and any } x \in K, \text{ the following } \frac{(T_j L_n)(x)}{n^{\alpha_j}} \leq k_j, \quad (6)$$

hold, for $j \in \{s, s+2\}$. Then, the convergence expressed by (5) is uniform on K and

$$n^{s-\alpha_s} \left| (L_n f)(x) - \sum_{i=0}^s \frac{f^{(i)}(x)}{n^i \cdot i!} (T_i L_n)(x) \right| \leq \frac{1}{s!} (k_s + k_{s+2}) \omega_1 \left(f^{(s)}; \frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}} \right), \quad (7)$$

for any $x \in K$, $n \geq n(s)$, where $\omega_1(f; \delta)$ denotes the modulus of continuity [1], (the first order modulus of smoothness) of function f .

2. THE SZÁSZ-MIRAKJAN-KANTOROVICH OPERATORS

In 1941, G. M. Mirakjan [2] defined the operators $S_n : C_2([0, +\infty)) \rightarrow C([0, +\infty))$,

given by $(S_n f)(x) = e^{-nx} \sum_{i=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$, for any $x \in [0, +\infty)$ and any $n \in \mathbb{N}$, where:

$$C_2([0, +\infty)) = \left\{ f \in C([0, +\infty)) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

The operators $(S_n)_{n \in \mathbb{N}}$ are named Mirakjan-Favard-Szász operators. They were intensively studied in 1944 by J. Favard [3] and in 1950 by O. Szász [4]. The Szász-Mirakjan-Kantorovich operators were first introduced by P. L. Butzer [5], in order to obtain an approximation process for spaces of integrable functions, on unbounded intervals. We recall that, P. L. Butzer defined the operators $K_n : L_1([0, +\infty)) \rightarrow B([0, +\infty))$, given by:

$$(K_n f)(x) = n \cdot e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \quad (8)$$

for any $x \in [0, +\infty)$ and any $n \in \mathbb{N}$.

Remark 3. The Szász-Mirakjan-Kantorovich operators are linear and positive.

More results and some interesting approximation properties for the Szász-Mirakjan-Kantorovich operators, defined at (8) may be found in [6] and [7]. The images of test functions by the operators Szász-Mirakjan-Kantorovich are given in the following:

Lemma 4. ([6]) Let $e_i(x) = x^i$, $i \in \{0, 1, 2, 3, 4\}$ be the test functions. The Szász-Mirakjan-Kantorovich operators satisfy:

$$\begin{aligned} (K_n e_0)(x) &= 1, & (K_n e_1)(x) &= x + \frac{1}{2n}, & (K_n e_2)(x) &= x^2 + \frac{2x}{n} + \frac{1}{3n^2}, \\ (K_n e_3)(x) &= x^3 + \frac{9x^2}{2n} + \frac{7x}{2n^2} + \frac{1}{4n^3}, & (K_n e_4)(x) &= x^4 + \frac{8x^3}{n} + \frac{15x^2}{n^2} + \frac{6x}{n^3} + \frac{1}{5n^4}, \end{aligned}$$

for any $x \in [0, +\infty)$ and any $n \in \mathbb{N}$.

3. MAIN RESULTS

Lemma 5. For any $n \in \mathbb{N}$ and any $x \in [0, +\infty)$, the following identities

$$(T_0 K_n)(x) = 1, \quad (9)$$

$$(T_1 K_n)(x) = \frac{1}{2}, \quad (10)$$

$$(T_2 K_n)(x) = n \left(x + \frac{1}{3n} \right), \quad (11)$$

$$(T_4 K_n)(x) = n^2 \left(3x^2 + \frac{5x}{n} + \frac{1}{5n^2} \right) \quad (12)$$

hold.

Proof. Taking relation (2) and Lemma 4 into account, we get

$$\begin{aligned} (T_0 K_n)(x) &= (K_n e_0)(x) = 1, \\ (T_1 K_n)(x) &= n(K_n \psi_x)(x) = n((K_n e_1)(x) - x(K_n e_0)(x)) = n \left(x + \frac{1}{2n} - x \right) = \frac{1}{2}, \\ (T_2 K_n)(x) &= n^2(K_n \psi_x^2)(x) = n^2 \left((K_n e_2)(x) - 2x(K_n e_1)(x) + x^2(K_n e_0)(x) \right) \\ &= n^2 \left(x^2 + \frac{2x}{n} + \frac{1}{3n^2} - 2x^2 - \frac{x}{n} + x^2 \right) = n \left(x + \frac{1}{3n} \right), \\ (T_4 K_n)(x) &= n^4(K_n \psi_x^4)(x) \\ &= n^4 \left((K_n e_4)(x) - 4x(K_n e_3)(x) + 6x^2(K_n e_2)(x) - 4x^3(K_n e_1)(x) + x^4(K_n e_0)(x) \right) \\ &= n^2 \left(3x^2 + \frac{5x}{n} + \frac{1}{5n^2} \right). \end{aligned}$$

Theorem 6. The following relations

$$\lim_{n \rightarrow \infty} (T_0 K_n)(x) = 1, \quad (13)$$

$$\lim_{n \rightarrow \infty} \frac{(T_2 K_n)(x)}{n} = x, \quad (14)$$

$$\lim_{n \rightarrow \infty} \frac{(T_4 K_n)(x)}{n^2} = 3x^2 \quad (15)$$

hold, for any $x \in [0, +\infty)$ and $n_0 \in \mathbb{N}$ exists, so that

$$(T_0 K_n)(x) = 1 = k_0, \quad (16)$$

$$\frac{(T_2 K_n)(x)}{n} \leq b + 1 = k_2, \quad (17)$$

$$\frac{(T_4 K_n)(x)}{n^2} \leq 3b^2 + 1 = k_4, \quad (18)$$

for any $x \in K = [0, b]$, $b > 0$ and any $n \in \mathbb{N}$, $n \geq n_0$.

Proof. The relations (13)-(15) follow immediately from Lemma 5 and (16)-(18) yield from (13)-(15), by taking the definition of the limit into account.

Now I suppose that $I = J = [0, +\infty)$, $E(I) = L_1([0, +\infty))$, $F(J) = B([0, +\infty))$, the functions $\varphi_{n,k}(x) = n \cdot e^{-nx} \frac{(nx)^k}{k!}$ be defined for any $x \in [0, +\infty)$, any $n, k \in \mathbb{N}_0$, $n \neq 0$ and the

functional $A_{n,k}(f) = \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt$ be defined for any $n, k \in \mathbb{N}_0$, $n \neq 0$.

So, I get the Szász-Mirakjan-Kantorovich operators.

Theorem 7. Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a function. If $x \in [0, +\infty)$, f is s times differentiable in a neighborhood of x , with $f^{(s)}$ continuous in x , then: $\lim_{n \rightarrow \infty} (K_n f)(x) = f(x)$, (19)

if $s = 0$ and $\lim_{n \rightarrow \infty} n \left((K_n f)(x) - f(x) - \frac{1}{2n} f'(x) \right) = \frac{1}{2} x f''(x)$, (20)

if $s = 2$.

If f is s times differentiable on $[0, +\infty)$, with $f^{(s)}$ continuous on $[0, +\infty)$, then the convergence from the relations (19) and (20) is uniform on any compact interval $K = [0, b] \subset [0, +\infty[$.

Moreover, the following: $|(K_n f)(x) - f(x)| \leq (1 + k_2) \omega_1 \left(f; \frac{1}{\sqrt{n}} \right)$ (21)

holds, for any $f \in L_1([0, +\infty))$, any $x \in K$ and any $n \in \mathbb{N}$, $n \geq n_0$.

Proof. It results from Theorem 2, with $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$, Lemma 5 and Theorem 6.

Remark 8. The identity (20) represents a relation of Voronovskaja's type, for the Szász-Mirakjan-Kantorovich operators.

Remark 9. The uniform convergence and the order of approximation for the Szász-Mirakjan-Kantorovich operators are proved without Bohman-Korovkin's and Shisha-Mond's Theorem.

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