ABOUT A TECHNIQUE OF SOLVING SOME DIFFERENTIAL STOCHASTIC ITÔ EQUATIONS

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Abstract. *The differential stochastic systems modeled the evolutive phenomena of environment influenced by stochastic forces. In this article it solved, using the Itô's formula, some differential stochastic systems for a vibrating string subject to a stochastic force and electric circuit.*

Keywords: *Itô equation, system, Brownian motion, vector.*

1. INTRODUCTION

The differential and integral stochastic calculation has developed over the last years due to the necessity of modulating growth phenomena through a probabilistic way of approach where the "noisy" environment in which these phenomena occurs has its own importance.

2. TABLE OF APPLICATION

Example 1: Let be the 2-dimensional stochastic differential equation:

$$dx_1 = x_2(t)dt + \alpha dB_1(t) \qquad dx_2 = x_1(t)dt + \alpha dB_2(t) \tag{1}$$

where (B_1(t), B_2(t)) is 2-dimensional Brownian motion and α,β are constants. With the notations:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ k = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},\ B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix} \tag{2}$$

the 2-dimensional stochastic differential equation (1) rewrite for a matriceal equation form:

$$dx(t) = Ax(t)dt + KdB_t \tag{3}$$

Statement 1 The solution of equation (3) is:

$$x(t) = \left(x(0) - KB(0)\right)\exp(At) + AK\exp(At)\int_0^t \left[\exp(-As)B_s ds + KBB_t\right] \tag{4}$$

Itô (4')

$$Var\left[x(t)\right] = K^2 t \tag{4''}$$

Proof: For solving using the integrating factor technique, the equation (3) is multiplied with exp(-At) and obtain:

$$\exp(-At)dx(t) - Ax(t)\exp(-At)dt = K\exp(-At)dB_t \tag{5}$$

The left hand side is the differential of the product exp(-At)x(t):

$$d\left(\exp(-At)x(t)\right) = K\exp(-At)dB_t \tag{5'}$$

and following:

$$\exp(-At)x(t) - x(0) = K\int_0^t \exp(-As)dB_s \tag{6}$$

Applied the Itô's formula for the integration by parts:

$$x(t) = x(0)\exp(At) + K\exp(At)\big(B_t\exp(At) - B_0\big) + \int_0^t \exp(-As)B_s ds \tag{7}$$

which is equivalent with equation (4).

For (4') we apply on the equation (6) the properties of the Itô integral and of the expectation, such that

$$E\left[x(t)\right] = \left(x(0) - KB_0\right)\exp(At)$$

For variance's equations we calculate the expression:

$$\left(x(t) - E\left[x(t)\right]\right)^2 = \left(AK\exp(At)\int_0^t \exp(-As)B_s ds + KB_t\right)^2$$

and using the properties of Brownian motion and of the expectation,

$$\left(x(t) - E\left[x(t)\right]\right)^2 =$$

$$= A^2K^2\exp(2At)\left[\int_0^t \exp(-As)B_s ds\right]^2 + K^2\left(B_t\right)^2 + +2AK^2Bt + \exp(At)\int_0^t \exp(-As)B_s ds + +KB_t$$

So, we get (4").

Example 2: Another example the differential stochastic system is the following: the charge Q(t) at time t at a fixed point in an electric circuit satisfies the differential equation:

$$\begin{aligned} &LQ''(t) + RQ'(t) + \left(\frac{1}{c}\right)Q(t) = F(t) \\ &Q(0) = Q_0 \\ &Q'(0) = I_0 \end{aligned} \tag{8}$$

where L is inductance, R is resistance, C is capacitance and F(t) the potential source at time t.

We may have a situation where some the coefficients, F(t), are not deterministic, it is a stochastic force,

$$F(t) = G_t + \alpha W_t$$

We introduce the vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} Q(t) \\ Q'(t) \end{pmatrix} \tag{9}$$

and obtain the following two equations:

$$x_1'(t) = x_2(t)$$

$$Lx_2'(t) = -Rx_2(t) - \left(\frac{1}{c}\right)x_1(t) + G_t + \alpha W_t$$

more so, in matrix notation,

$$dx(t) = Ax(t)dt + H(t)dt + KdB_t$$

(10)

where $dx(t) = \begin{pmatrix} dx_1(t) \\ dx_1(t) \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, H(t) = \begin{pmatrix} 0 \\ \frac{1}{L}G_t \end{pmatrix}, K = \begin{pmatrix} 0 \\ \frac{a}{L} \end{pmatrix}$ and B_t is a 1-dimensional Brownian motion.

Statement 2: The solution of equation (10) is:

$$x(t) = \left(x(0) - KB(0)\right)\exp(At) + \exp(At)\int_0^t \exp(-As)\left[H(s) + AKB_s\right]ds + KB_t \quad (11)$$

$$\exp(-As)\left[H(s) + AKB_s\right]ds + KBt\int_0^t \exp(-As)H(s)\,ds$$

(11')

$$Var\left(x(t)\right) = K^2 t \quad (11")$$

Proof: Similar, the equation (10) is multiplied with the integrating factor, *exp(-At)* and obtain:

$$\exp(-At)dx(t) - Ax(t)\exp(-At)dt = H(t)\exp(-At)dt + K\exp(-At)dB_t \quad (12)$$

The left hand side is the differential of the product *exp(-At)x(t)*:

$$d\left(\exp(-At)x(t)\right) = H(t)\exp(-At)dt + K\exp(-At)dB_t \quad (12')$$

and following:

$$\exp(-At)x(t) - x(0) = \int_0^t H(s)\exp(-As)ds + \int_0^t \exp(-As)dB_s \quad (13)$$

Appling Itô's formula for the integration by parts:

$$x(t) = x(0)\exp(At)\int_0^t H(s)\exp(-As)ds + K\exp(At)\left(B_t\exp(-At) - B_0 + A\int_0^t H(s)\exp(-As)ds\right) \quad (14)$$

which is equivalent with equation (11).

For (11') we apply on the equation (14) the properties of the Itô integral and of the expectation, such that:

$$E\left[x(t)\right] = \left(x(0) - KB_0\right)\exp(At) + \exp(At)\int_0^t \exp(-As)H(s)\,ds$$

For variance's equations on evaluate the expression:

$$\left(x(t) - E\left[x(t)\right]\right)^2 = \left(AK\exp(At)\int_0^t \exp(-As)B_s ds + KB_t\right)^2$$

and using the properties of Brownian motion and of the expectation, again:

$$\left(x(t) - E\left[x(t)\right]\right)^2 =$$

$$= A^2K^2\exp(2At)\left[\int_0^t \exp(-As)B_s ds\right]^2 + +K^2\left(B_t\right)^2 + 2AK^2B_t\exp(At)\int_0^t \exp(-As)B_s ds + +KB_t$$

Thus obtaining (11").

3. CONCLUSIONS

Using the stochastic differential calculus Itô, the integrant factor technique, we have determined the solutions of two matrix differential equations. We have also evaluated the expected value and variance of the solutions given.

The first example is a model for a vibrating string subject to a stochastic force; the stochastic differential matrix equation is from type (3)

$dx(t) = Ax(t)dt + KdB_t$

$k = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$ where $(B_1(t),B_2(t))$ is 2-dimensional Brownian motion and α, β are constants. The solution is:

$$x(t) = \left(x(0) - KB(0)\right)\exp(At) + AK\exp(At)\int_0^t \exp(-As)B_s ds + KB_t$$

$$E\left[x(t)\right] = \left(x(0) - KB_0\right)\exp(At),\ Var\left[x(t)\right] = K^2 t$$

or, if the stochastic differential matrix equation is from type (10), a model for electric circuit

$$dx(t) = Ax(t) + H(t)dt + KdB_t$$

then the solution is:

$$x(t) = \left(x(0) - KB(0)\right)\exp(At) + \exp(At)\int_0^t \exp(-As)\left[H(s) + AKB_s\right]ds + KB_t$$

$$E\left[x(t)\right] = \left(x(0) - KB_0\right)\exp(At) + \exp(At)\int_0^t \exp(-As)H(s)ds,\ Var\left[x(t)\right] = K^2 t$$

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