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THE HYERS-ULAM STABILITY OF SOME FUNCTIONAL EQUATIONS DEFINED BY MEANS

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Abstract. *We prove the Hyers-Ulam stability of some functional equations involving function that transform some classical means in another classical means.*

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1. INTRODUCTION

In [1] we have studied the stability of some functional equations that keep classical means, by using the stability of Jensen equation. Some of the results obtained in [1] will be used in the present paper.

Let $I \subset \mathbb{R}$ be an interval and $m : I \times I \to I$ a function with the property:

$$\min\{x, y\} \le m(x, y) \le \max\{x, y\}, \text{ for all } x, y \in I.$$

Such a function is called mean on I.

In this paper we use the classical means:

$$\begin{aligned} m_a(x,y) &= \frac{x+y}{2} \quad \text{(arithmetic mean)}, \ x, y \in \mathbb{R} \\ m_g(x,y) &= \sqrt{x \cdot y} \quad \text{(geometric mean)}, \ x, y \ge 0 \\ m_h(x,y) &= \frac{2xy}{x+y} \quad \text{(harmonic mean)}, \ x, y > 0. \end{aligned}$$

Let $m_1 : A_1 \times A_1 \to A_1$ be a mean of A_1, $m_2 : A_2 \times A_2 \to A_2$ be a mean of A_2 and $F : A_1 \to A_2$ be a function.

Definition 1.1. *We say that the function $F : A_1 \to A_2$ transforms the mean m_1 in the mean m_2 if the following relation holds:*

$$F(m_1(x,y)) = m_2(F(x), F(y)), \text{ for all } x, y \in A_1.$$

If $A_1 = A_2$ and $m_1 = m_2$ we say that the function F is m invariant $\square$.

Remark 1.1. a) The function $F : \mathbb{R} \to \mathbb{R}$ is m_a-invariant if and only if the function F verifies the functional equation:

$$\text{(J)} \qquad F\left(\frac{x+y}{2}\right) = \frac{F(x)+F(y)}{2}, \text{ for all } x, y \in \mathbb{R},$$

that is the Jensen functional equation.

Thus a function $F : \mathbb{R} \to \mathbb{R}$ is m_a-invariant iff it is a Jensen function.

b) The case $m_1 = m_2$ was studied in [1] for some particular equations.

c) Equations of the form $f(F(x,y)) = G(f(x), f(y))$ for F and G given functions and the unknown function f was considered for the first time by J.Aczél in [2] $\Box$.

We recall some known stability results for Jensen functional equation from [3, 4, 5]

Theorem 1.1 [3, 4]. *The functions $F : \mathbb{R} \to \mathbb{R}$ satisfying Jensen functional equation (J) are of the form*

$$F(x) = A(x) + b, \quad x \in \mathbb{R},$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function ($A(x+y) = A(x) + A(y),$, $x, y \in \mathbb{R}$) and b is a real constant.

Theorem 1.2 [3, 4]. *The continuous functions $F : \mathbb{R} \to \mathbb{R}$ satisfying Jensen equation (J) are:*

$$F(x) = ax + b, \quad x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$.

Theorem 1.3 [5]. (Hyers-Ulam stability of Jensen equation) *For every $\varepsilon > 0$ and for every function $f : \mathbb{R} \to \mathbb{R}$ satisfying the inequality:*

$$\left| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right| \leq \varepsilon, \text{ for all } x, y \in \mathbb{R},$$

there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ satisfying Jensen equation such that:

$$|F(x) - f(x)| \leq 2\varepsilon, \text{ for all } x \in \mathbb{R}.$$

Remark 1.2 [5].

• The function F from Theorem 1.3 is defined by:

$$F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} + f(0).$$

• The function A from Theorem 1.1 with the property

$$F(x) = A(x) + b$$

is given by $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ and $b = f(0)$

• If in Theorem 1.3 the function f is continuous then

$$F(x) = ax + b, \ x \in \mathbb{R},$$

where $a = \lim_{n\to\infty} \frac{f(2^n)}{2^n}$ and $b = f(0)$.

Theorem 1.4. (Kominek [6]) *Let $D \subset \mathbb{R}^n$ be a subset with non-empty interior and Y be a real Banach space. Assume that there exists an x_0 in the interior of D such that $D_0 = D - x_0$ fulfills the condition $\frac{D}{2} \subset D_0$. Let a mapping $f : D \to Y$ satisfy the inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all$x, y \in D$. Then there exist a mapping $F : \mathbb{R}^n \to Y$ and a constant $k > 0$ such that:

$$2F\left(\frac{x+y}{2}\right) = F(x) + F(y), \text{ for all } x, y \in \mathbb{R}^n$$

and

$$\|f(x) - F(x)\| \leq k, \quad x \in D.$$

Let (m_1, m_2)be the functional equation from the Definition 1.1:

$$(m_1, m_2): \quad \begin{cases} F : A_1 \to A_2 \\ F(m_1(x, y)) = m_2(F(x), F(y)), \ x, y \in A_1. \end{cases}$$

The notion of Hyers-Ulam stability introduced in [7] for the Cauchy functional equation and extended in [5], for the equation (m_1, m_2) is defined below:

Definition 1.2. *We say that the functional equation* (m_1, m_2) *is stable if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every function* $g : A_1 \to A_2$ *that verifies the inequality:*

$$|g(m_1(x, y)) - m_2(g(x), g(y))| \leq \varepsilon, \text{ for all } x, y \in A_1,$$

there exists a function $G : A_1 \to A_2$ *satisfying the equation* (m_1, m_2) *such that:*

$$|G(x) - g(x)| \leq \delta, \text{ for all } x \in A_1. \quad \square$$

If $m_1 = m_2 = m_a$ is the arithmetic mean then from Theorem 1.3 it follows that the equation (m_a, m_a) is stable and from Theorem 1.1 and Theorem 1.2 we have the explicit form of Jensen function G which approximate the function g.

Next we will prove that the equations (m_g, m_a) and(m_h, m_a) are stable.

2. THE STABILITY OF THE EQUATION (m_g, m_a).

The equation (m_g, m_a) is:

$$(m_g, m_a): \quad \begin{cases} F : (0, \infty) \to \mathbb{R} \\ F(\sqrt{xy}) = \dfrac{F(x) + F(y)}{2}, \ x > 0, \ y > 0. \end{cases}$$

Theorem 2.1. *The function* $F : (0, \infty) \to \mathbb{R}$ *verifies the equation* (m_g, m_a) *iff there exists a unique additive function* $A : \mathbb{R} \to \mathbb{R}$ *and a constant* $b \in \mathbb{R}$ *such that*

$$F(x) = A(\log x) + b, \quad x \in (0, \infty).$$

If F *is continuous then* $F(x) = a \log x + b$, $x > 0$, *where* a, b *are constant real numbers.*

Proof. In the equation $F(\sqrt{xy}) = \dfrac{F(x) + F(y)}{2}$, $x > 0$, $y > 0$ we put $x = e^u$, $y = e^v$, $u, v \in \mathbb{R}$ and we obtain the equation:

$$F(e^{\frac{u+v}{2}}) = \frac{F(e^u) + F(e^v)}{2}, \quad u, v \in \mathbb{R}.$$

If we denote $G(u) = F(e^u)$, $u \in \mathbb{R}$, the function $G : \mathbb{R} \to \mathbb{R}$ verifies Jensen equation:

$$G\left(\frac{u+v}{2}\right)=\frac{G(u)+G(v)}{2}, \quad u,v\in\mathbb{R}.$$

From Theorem 1.1 there exists a unique additive function $A:\mathbb{R}\to\mathbb{R}$ and a constant $b\in\mathbb{R}$ such that

$$G(u)=A(u)+b, \quad u\in\mathbb{R}.$$

By the relation $F(x)=G(\log x)$ follows

$$F(x)=A(\log x)+b, \quad x>0.$$

If $F$is continuous then G is continuous and from Theorem 1.2 we have

$$G(u)=au+b,\ u\in\mathbb{R} \text{ and } F(x)=a\log x+b,\ x>0.$$

Theorem 2.2. *If $f:(0,\infty)\to\mathbb{R}$ is a function with the property:*

$$\left|f(\sqrt{xy})-\frac{f(x)+f(y)}{2}\right|\le\varepsilon$$

then there exists a unique additive function $A:\mathbb{R}\to\mathbb{R}$, a constant $b\in\mathbb{R}$ such that

$$|f(x)-A(\log x)-b|\le 2\varepsilon, \text{ for all } x>0.$$

Moreover if f is continuous then there exist real constants a,b uniquely determined such that

$$|f(x)-a\log x-b|\le 2\varepsilon, \text{ for all } x>0.$$

Proof. In the inequality

$$\left|f(\sqrt{xy})-\frac{f(x)+f(y)}{2}\right|\le\varepsilon,$$

make the substitutions $x=e^u$, $y=e^v$, $f(e^u)=g(u)$ and we get for the function $g:\mathbb{R}\to\mathbb{R}$ the inequality

$$\left|g\left(\frac{u+v}{2}\right)-\frac{g(u)+g(v)}{2}\right|\le\varepsilon, \quad u,v\in\mathbb{R}.$$

In view of Theorem 1.3, there exists a unique function $G:\mathbb{R}\to\mathbb{R}$ satisfying Jensen equation and

$$|g(u)-G(u)|\le 2\varepsilon, \quad u\in\mathbb{R}.$$

Replacing $u=\log x$ and $F(x)=G(\log x)$ we get in view of Theorem 2.1,

$$F(x)=A(\log x)+b, \quad x>0,$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function.

If f is continuous, then F is continuous too, therefore

$$F(x)=a\log x+b,\quad x>0,\ a,b\in\mathbb{R}.$$

3. THE STABILITY OF THE EQUATION (m_h,m_a) .

The equation (m_h,m_a) is:

$$(m_h,m_a):\quad \begin{cases} F:(0,\infty)\to\mathbb{R} \\ F\left(\dfrac{2xy}{x+y}\right)=\dfrac{F(x)+F(y)}{2},\ x>0,\ y>0. \end{cases}$$

Theorem 3.1. *The function* $F:(0,\infty)\to\mathbb{R}$ *verifies the equation* (m_h,m_a) *if there exists a unique additive function* $A:\mathbb{R}\to\mathbb{R}$ *and a constant* $b\in\mathbb{R}$ *such that*

$$F(x)=A\left(\frac{1}{x}\right)+b,\quad x>0.$$

If F *is continuous then*$F(x)=\dfrac{a}{x}+b$, $x>0$, *where* $a,b\in\mathbb{R}$.

Proof. With the substitution $G(x)=F\left(\dfrac{1}{x}\right)$, we get for the function$G:(0,\infty)\to\mathbb{R}$, Jensen equation on $(0,\infty)$

$$G\left(\frac{x+y}{2}\right)=\frac{G(x)+G(y)}{2},\quad x,y\in(0,\infty).$$

Taking account of a result from [4], if $C\subset\mathbb{R}^n$ is a convex set such that the interior of $C\neq\emptyset$ and $G:C\to\mathbb{R}$ is a solution of Jensen equation on C, then G is given by$G(x)=A(x)+b$, $x\in C$, where $A:\mathbb{R}^n\to\mathbb{R}$ is an additive function and b is a constant. Moreover, from [3] if $G:I\to\mathbb{R}$ is continuous and verifies Jensen equation, then

$$G(x)=ax+b,\ x\in I,\ \text{where } I\subset\mathbb{R} \text{ is an interval.}$$

It follows that $G(x)=A(x)+b$, $x>0$, where $A:\mathbb{R}\to\mathbb{R}$ is an additive function and b is a constant.

Thus $F(x)=A\left(\dfrac{1}{x}\right)+b$, $x>0$ and if F is continuous then

$$F(x)=ax+b,\quad x>0.$$

Theorem 3.2. *If* $f:(0,\infty)\to\mathbb{R}$ *is a function with the property*

$$\left|f\left(\frac{2xy}{x+y}\right)-\frac{f(x)+f(y)}{2}\right|\le\varepsilon,\quad x>0,\ y>0,$$

then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$*, a constant* $b \in \mathbb{R}$ *such that:*

$$\left| f(x) - A\left(\frac{1}{x}\right) - b \right| \le 2\varepsilon, \quad x > 0.$$

Moreover if f *is continuous then there exists unique real constant* a*,* b *such that:*

$$\left| f(x) - \frac{a}{x} - b \right| \le 2\varepsilon, \quad x > 0.$$

Proof. In the inequality

$$\left| f\left(\frac{2xy}{x+y}\right) - \frac{f(x)+f(y)}{2} \right| \le \varepsilon$$

we make the substitution $g(x) = f\left(\frac{1}{x}\right)$ and $\frac{1}{x} = u$, $\frac{1}{y} = v$ and we obtain for the function $g : (0,\infty) \to \mathbb{R}$ the inequality

$$\left| g\left(\frac{u+v}{2}\right) - \frac{g(u)+g(v)}{2} \right| \le \varepsilon, \quad u, v \in (0,\infty).$$

Using the stability of Jensen functional equation (Theorem 1.3, Theorem 1.2) and Theorem 1.4 for $D = (0,\infty)$ and $x_0 = 1$, we obtain the conclusion.

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