ORIGINAL PAPER

A NEW GEOMETRIC INEQUALITY AND ITS APPLICATIONS

JIAN LIU

East China Jiaotong University, 330013, Nanchang City, Jiangxi Province, China

Abstract. *In this paper, we establish a ternary quadratic geometric inequality involving a triangle and a point by using the polar moment of the inertia inequality of M.S. Klamkin. We also give new result some applications, and propose several related conjectures which are checked by the computer.*

Keywords: *triangle, point, the polar moment of the inertia inequality, real number, inequality.*

1. INTRODUCTION

In 1975, M. S. Klamkin [1] established the following geometric inequality: Let ABC be a triangle with sides $a = BC$, $b = CA$, $c = AB$, and let P be an arbitrary point, the distances of P from the vertices A, B, C are R_1, R_2, R_3 respectively. Then the inequality

$$(x+y+z)\left(xR_1^2+yR_2^2+zR_3^2\right)\geq yza^2+zxb^2+xyc^2, \tag{1}$$

holds for all real numbers x, y, z. Equality if and only if P lies the plane of ΔABC and $x:y:z=\vec{S}_{\triangle PBC}:\vec{S}_{\triangle PCA}:\vec{S}_{\triangle PAB}$ ($\vec{S}_{\triangle PBC}$ denotes the algebra area etc.)

Inequality (1) is called the polar moment of the inertia inequality by M. S. Kalmkin. It is one of the most important results of geometric inequalities for the triangle, and there exist many consequences and applications for it, e.g., see [1-6]. The author has also researched this inequality. In 1992, we found it can be deduced the weighted inequality for the sides a, b, c of the triangle (see [5]):

$$\frac{s-a}{x}+\frac{s-b}{y}+\frac{s-c}{z}\geq\frac{(xa+yb+zc)s}{yza+zxb+xyc}, \tag{2}$$

where $s=(a+b+c)/2$ and x, y, z are positive real numbers. Equality holds if and only if $x=y=z$. The inequality (2) was used to prove several geometric inequalities which connect with a point of a triangle in [5]. In 2007, the author [6] applied Klamkin inequality (1) and inversion transformation to establish the following geometric inequality with positive real numbers x, y, z:

$$\frac{R_1^2}{x}+\frac{R_2^2}{y}+\frac{R_3^2}{z}\geq\frac{aR_1+bR_2+cR_3}{\sqrt{yz+zx+xy}}. \tag{3}$$

Equality holds if and only if ΔABC is acute-angled, P coincides with its orthocenter and $x:y:z=\cot A:\cot B:\cot C$. The author also gave inequality (3) some applications in [6].
The purpose of this paper is to establish the following ternary quadratic geometric inequality by using Klamkin's polar moment of the inertia inequality:
Theorem: For any arbitrary point P and real numbers x, y, z, we have

$$\frac{R_2^2+R_2^3}{a^2}x^2+\frac{R_3^2+R_1^2}{b^2}y^2+\frac{R_1^2+R_2^2}{c^2}z^2\geq\frac{2}{3}(yz+zx+xy), \tag{4}$$

Corresponding author: china99jian@163.com

with equality if and only if ΔABC is equilateral and P is its center.

In the next section we will prove the theorem. In the third section, we will give some applications of inequality (4). In the last section, several related conjectures are put forward.

2. THE PROOF OF THE THEOREM

Proof. To simplify matters. We denote cyclic sum by Σ, then inequality (4) is

$$\sum\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)R_1^2 \ge \frac{2}{3}\sum yz . \tag{5}$$

According to Klamkin inequality (1), we have

$$2\sum\frac{x^2}{a^2}\sum\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)R_1^2 \ge \sum\left(\frac{z^2}{c^2}+\frac{x^2}{a^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)a^2 \tag{6}$$

If $\sum yz < 0$, then (5) holds clearly. If $\sum yz > 0$, from inequality (6), in order to prove (5) we need to prove that

$$\sum\left(\frac{z^2}{c^2}+\frac{x^2}{a^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)a^2 \ge \frac{4}{3}\sum yz\sum\frac{x^2}{a^2}. \tag{7}$$

By replacing $x \to xa, y \to yb, z \to zc$, the above inequality is equivalent to

$$\sum\left(z^2+x^2\right)\left(x^2+y^2\right)a^2 \ge \frac{4}{3}\sum yzbc\sum x^2 , \tag{8}$$

Namely

$$3\sum\left(z^2+x^2\right)\left(x^2+y^2\right)a^2 - 4\sum yzbc\sum x^2 \ge 0, \tag{9}$$

That is

$$a_1a^2+b_1a+c_1 \ge 0, \tag{10}$$

where

$$a_1 = 3\left(z^2+x^2\right)\left(x^2+y^2\right)$$
$$b_1 = -4\left(zc+yb\right)x,$$
$$c_1 = 3\left(y^2+z^2\right)\left[\left(x^2+y^2\right)b^2+\left(z^2+x^2\right)c^2\right]-4yzbc\left(x^2+y^2+z^2\right).$$

We rewrite c_1 as follow:

$$c_1 = \left[3\left(b^2+c^2\right)\left(y^2+z^2\right)-4yzbc\right]x^2+\left(y^2+z^2\right)\left(3b^2y^2+3c^2z^2-4yzbc\right).$$

So, it is easy to see that $c_1 > 0$. Thus to prove (10) we need to show that $4a_1c_1-b_1^2 \ge 0$. However, we can obtain the following identity:

$$4a_1c_1 - b_1^2 = a_2b^2 + b_2b + c_2, \tag{11}$$

where

$$a_2 = 20x^6y^2 + 36x^6z^2 + 40x^4y^4 + 76x^4y^2z^2 + 36x^4z^4 + 20x^2y^6$$
$$+76x^2y^4z^2 + 56x^2y^2z^4 + 36y^6z^2 + 36y^4z^4$$
$$b_2 = -16c\left(5x^4 + 5x^2y^2 + 5x^2z^2 + 3y^2z^2\right)\left(x^2 + y^2 + z^2\right)yz,$$
$$c_2 = 4c^2(9x^6y^2 + 5x^6z^2 + 9x^4y^4 + 19x^4y^2z^2 + 10x^4z^4 + 14x^2y^4z^2$$
$$+19x^2y^2z^4 + 5x^2z^6 + 9y^4z^4 + 9y^2z^6).$$

Since again $a_2 > 0, c_2 > 0$, it sufficies to prove that $4a_2c_2 - b_2^2 \geq 0$. But

$$4a_2c_2 - b_2^2 =$$
$$= 192c^2\left(z^2 + x^2\right)\left(x^2 + y^2\right)\cdot \tag{12}$$
$$\left[30\sum y^6z^6 + 15\sum y^4z^4\left(y^4 + z^4\right) - 18x^2y^2z^2\sum\left(y^2 + z^2\right)x^4 + 2x^2y^2z^2\sum x^6 - 78x^4y^4z^4\right]$$

Hence, we only need to prove the following inequality:

$$30\sum v^3w^3 + 15\sum v^2w^2\left(v^2 + w^2\right) - 18uvw\sum\left(v + w\right)u^2 + 2uvw\sum u^3 - 78u^2v^2w^2 \geq 0 \tag{13}$$

holds for non-negative real numbers *u, v, w*. Denote the left hand side of (13) by *Q*, after analyzing we obtain the following identity:

$$Q = Q_1 + Q_2 + Q_3 \tag{14}$$

where

$$Q_1 = 15\sum v^2w^2\left(v - w\right)^2 + 12uvw\sum u\left(v - w\right)^2$$
$$Q_2 = 2uvw\left(\sum u^3 - 3uvw\right) + 30\left(\sum v^3w^3 - 3u^2v^2w^2\right)$$
$$Q_3 = 30\sum v^2w^2\left(u - v\right)\left(u - w\right)$$

Obviously, inequalities $Q_1 \geq 0,\ Q_2 \geq 0$ hold true. In addition, by replacing

$$u \to vw,\ v \to wu,\ w \to uv$$

in the simple case of the famous Schur inequality

$$\sum u\left(u - v\right)\left(u - w\right) \geq 0, \tag{15}$$

we know $\sum v^2w^2\left(u - v\right)\left(u - w\right) \geq 0$, hence $Q_3 \geq 0$. Therefore, the inequality $Q \geq 0$ follows from (14).

It is easy to see that equality in (7) holds if and only if $x = y = z$ and triangle *ABC* is equilateral. Further, the equality in (4) holds just as the mentions of our theorem. This completes the proof of the Theorem. □

3. APPLICATIONS OF THE THEOREM

In this section, we will discuss some applications of our theorem.

In inequality (4), for $x=\frac{a}{\sqrt{R_2^2+R_3^2}}$, $y=\frac{b}{\sqrt{R_3^2+R_1^2}}$, $z=\frac{c}{\sqrt{R_1^2+R_2^2}}$, we obtain

Corollary 1: For *ΔABC* and arbitrary point P, we have

$$\frac{bc}{\sqrt{\left(R_3^2+R_1^2\right)\left(R_1^2+R_2^2\right)}}+\frac{ca}{\sqrt{\left(R_1^2+R_2^2\right)\left(R_2^2+R_3^2\right)}}+\frac{ab}{\sqrt{\left(R_2^2+R_3^2\right)\left(R_3^2+R_1^2\right)}}\le\frac{9}{2} \tag{16}$$

From inequality (16), we can easy get two following corollaries again:

Corollary 2: For *ΔABC* and arbitrary point *P*, we have

$$\left(R_2^2+R_3^2\right)\left(R_3^2+R_1^2\right)\left(R_1^2+R_2^2\right)\ge\frac{8}{27}(abc)^2 \tag{17}$$

Corollary 3: For *ΔABC* and arbitrary point *P*, we have

$$\frac{\sqrt{R_2^2+R_3^2}}{a}+\frac{\sqrt{R_3^2+R_1^2}}{b}+\frac{\sqrt{R_1^2+R_2^2}}{c}\ge\sqrt{6} \tag{18}$$

In 1996, the author [7] estabilished the following decline theorem about the ternary quadratic inequality: Let p_1, p_2, p_3, q_1, q_2, q_3 and m be positive real numbers. If the ternary quadratic inequality:

$$p_1^m x^2+p_2^m y^2+p_3^m z^2\ge q_1^m yz+q_2^m zx+q_3^m xy \tag{19}$$

holds for arbitrary real numbers *x, y, z*, then

$$p_1^k x^2+p_2^k y^2+p_3^k z^2\ge q_1^k yz+q_2^k zx+q_3^k xy \tag{20}$$

where $0<k<m$. By the decline theorem and inequality (4), we get the more general result than (18):

Corollary 4: For *ΔABC* and arbitrary point *P*, we have

$$\frac{\sqrt{R_2^2+R_3^2}}{a}x^2+\frac{\sqrt{R_3^2+R_1^2}}{b}y^2+\frac{\sqrt{R_1^2+R_2^2}}{c}z^2\ge\sqrt{\frac{2}{3}}\left(yz+zx+xy\right) \tag{21}$$

Putting in (21) $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, one has

Corollary 5: For *ΔABC* and arbitrary point *P*, we have

$$\sqrt{R_2^2+R_3^2}+\sqrt{R_3^2+R_1^2}+\sqrt{R_1^2+R_2^2}\geq\sqrt{\frac{2}{3}}\left(\sqrt{bc}+\sqrt{ca}+\sqrt{ab}\right) \tag{22}$$

Let Ω_1 denote one of the Crelle – Brocard point of *ΔABC*, then we have the following formula (see [4, P_{279}]):

$$A\Omega_1^2=\frac{c^4b^2}{b^2c^2+c^2a^2+a^2b^2} \tag{23}$$

So, if we order $P=\Omega_1$ in (4), then it follows at once

$$\left(c^2a^2+b^4\right)x^2+\left(a^2b^2+c^4\right)y^2+\left(b^2c^2+a^4\right)z^2\geq\frac{2}{3}(yz+zx+xy)\left(b^2c^2+c^2a^2+a^2b^2\right) \tag{24}$$

By replacing $x\to y,\ y\to z,\ z\to x$, we get the weighted inequality for the sides of *ΔABC*:

Corollary 6: For *ΔABC* and arbitrary numbers *x, y, z*, we have

$$\left(b^2c^2+a^4\right)x^2+\left(c^2a^2+b^4\right)y^2+\left(a^2b^2+c^4\right)z^2\geq\frac{2}{3}(yz+zx+xy)\left(b^2c^2+c^2a^2+a^2b^2\right) \tag{25}$$

We denote the sides $B'C',\ C'A',\ A'B'$ of *ΔA'B'C'* by *a', b', c'*, the distances form arbitrary point *P'* to the vertices *A', B', C'* by D_1', D_2', D_3' respectively. In (4) we take

$$x=\frac{D_1'}{a'},\ y=\frac{D_2'}{b'},\ z=\frac{D_3'}{c'}$$

then using Hayashi inequality [8] (it can be deduced by Klamkin inequality (1), see [4]):

$$\frac{D_2'D_3'}{b'c'}+\frac{D_3'D_1'}{c'a'}+\frac{D_1'D_2'}{a'b'}\geq 1 \tag{26}$$

we obtain

Corollary 7: For *ΔABC, ΔA'B'C'* and two points *P, P'*, we have

$$\frac{\left(R_2^2+R_3^2\right)D_1'^2}{(aa')^2}+\frac{\left(R_3^2+R_1^2\right)D_2'^2}{(bb')^2}+\frac{\left(R_1^2+R_2^2\right)D_3'^2}{(cc')^2}\geq\frac{2}{3} \tag{27}$$

If we order $\Delta A'B'C'\cong\Delta ABC$ and *P=P'*, then we get

Corollary 8: For *ΔABC* and arbitrary point *P*, we have

$$\frac{\left(R_2^2+R_3^2\right)R_1^2}{a^4}+\frac{\left(R_3^2+R_1^2\right)R_2^2}{b^4}+\frac{\left(R_1^2+R_2^2\right)R_3^2}{c^4}\geq\frac{2}{3} \tag{28}$$

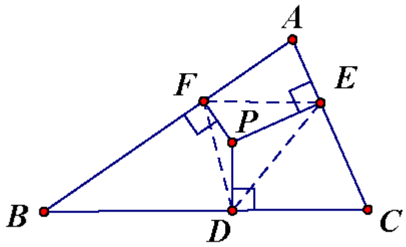

Fig. 1.

Let P lie in the plane of ΔABC, and let D, E, F be the orthogonal projections of P on the lines BC, CA, AB respectively (See Fig. 1). Put $PD = r_1$, $PE = r_2$, $PF = r_3$, note that $EF = R_1 \sin A$, $FD = R_2 \sin B$, $DE = R_3 \sin C$, inequality (4) is applied to pedal triangle DEF to give

Corollary 9: If P lies in the plane of $\triangle ABC$ and does not coincide with the vertices A, B, C, then the inequality

$$\frac{r_2^2+r_3^2}{R_1^2\sin^2 A}x^2+\frac{r_3^2+r_1^2}{R_2^2\sin^2 B}y^2+\frac{r_1^2+r_2^2}{R_3^2\sin^2 C}z^2\geq\frac{2}{3}\left(yz+zx+xy\right) \tag{29}$$

holds for arbitrary real numbers x, y, z.

Obviously, inequality (29) is equivalent to the following:

$$\frac{r_2^2+r_3^2}{R_1^2}x^2+\frac{r_3^2+r_1^2}{R_2^2}y^2+\frac{r_1^2+r_2^2}{R_3^2}z^2\geq\frac{2}{3}\left(yz\sin B\sin C+zx\sin C\sin A+xy\sin A\sin B\right) \tag{30}$$

Remark: For $x = y = z = 1$ in (29) we get

$$\frac{r_2^2+r_3^2}{R_1^2\sin^2 A}+\frac{r_3^2+r_1^2}{R_2^2\sin^2 B}+\frac{r_1^2+r_2^2}{R_3^2\sin^2 C}\geq 2 \tag{31}$$

When triangle ABC is acute-angled, the author [9] has generalized the above inequality to the case involving two triangles:

$$\frac{r_2^2+r_3^2}{R_1^2\sin^2 A'}+\frac{r_3^2+r_1^2}{R_2^2\sin^2 B'}+\frac{r_1^2+r_2^2}{R_3^2\sin^2 C'}\geq 2 \tag{32}$$

where A', B', C' are the angles of arbitrary triangle $A'B'C'$. Equality holds if and only if $\sin A:\sin^2 A'=\sin B:\sin^2 B'=\sin C:\sin^2 C'$ and P is the incenter of acute triangle ABC.

In (29), we take $x = a$, $y = b$, $z = c$. Then using $bc\sin B\sin C=h_a^2$ (h_a denotes the altitude of BC etc.), we get

Corollary 10: If P lies in the plane of $\triangle ABC$ and does not coincide with the vertices A, B, C, then

$$\frac{r_2^2+r_3^2}{R_1^2}a^2+\frac{r_3^2+r_1^2}{R_2^2}b^2+\frac{r_1^2+r_2^2}{R_3^2}c^2\geq\frac{2}{3}\left(h_a^2+h_b^2+h_c^2\right) \tag{33}$$

In addition, for $x = R_1, y = R_2, z = R_3$ in (29), it follows that

Corollary 11: For any ΔABC and arbitrary point P, we have

$$\frac{r_2^2+r_3^2}{\sin^2 A}+\frac{r_3^2+r_1^2}{\sin^2 B}+\frac{r_1^2+r_2^2}{\sin^2 C}\geq\frac{2}{3}\left(R_2R_3+R_3R_1+R_1R_2\right) \tag{34}$$

In (29) we take $x=\tan\frac{A}{2}$, $y=\tan\frac{B}{2}$, $z=\tan\frac{C}{2}$, then using well-known triangle identity:

$$\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1, \tag{35}$$

We get again the following:

Corollary 12: Let P be an arbitrary point which does not coincide with the vertices of ΔABC, then

$$\frac{r_2^2+r_3^2}{R_1^2\cos^4\frac{A}{2}}+\frac{r_3^2+r_1^2}{R_2^2\cos^4\frac{B}{2}}+\frac{r_1^2+r_2^2}{R_3^2\cos^4\frac{C}{2}}\geq\frac{8}{3} \tag{36}$$

In inequality (30) we put $x=\frac{R_1}{a^2}$, $y=\frac{R_2}{b^2}$, $z=\frac{R_3}{c^2}$, it follows that

$$\begin{aligned}&\frac{r_2^2+r_3^2}{a^4}+\frac{r_3^2+r_1^2}{b^4}+\frac{r_1^2+r_2^2}{c^4}\geq\frac{2}{3}\left(\frac{R_2R_3}{b^2c^2}\sin B\sin C+\frac{R_3R_1}{c^2a^2}\sin C\sin A+\frac{R_1R_2}{a^2b^2}\sin A\sin B\right)\\&=\frac{1}{6R^2}\left(\frac{R_2R_3}{bc}+\frac{R_3R_1}{ca}+\frac{R_1R_2}{ab}\right)\geq\frac{1}{6R^2}\end{aligned} \tag{37}$$

(where R is the radius of ΔABC) the last step was obtained by using Hayashi inequality (26) for the ΔABC and point P. Therefore, we have

Corollary 13: For any ΔABC and arbitrary point P in the plane of ΔABC, the following inequality holds:

$$\frac{r_2^2+r_3^2}{a^4}+\frac{r_3^2+r_1^2}{b^4}+\frac{r_1^2+r_2^2}{c^4}\geq\frac{1}{6R^2} \tag{37}$$

4. SEVERAL CONJECTURES

In the last section, we propose several related conjectures.

The author has considered the stronger inequalities of Corollary 5, 8 and 11. After being checked by the computer, the following three conjectures are put forward respectively:

Conjecture 1: For *ΔABC* and arbitrary point *P* holds:

$$\sqrt{R_2^2+R_3^2}+\sqrt{R_3^2+R_1^2}+\sqrt{R_1^2+R_2^2}\ge\sqrt{\frac{2}{3}}\left(a+b+c\right) \qquad (38)$$

Conjecture 2: For *ΔABC* and arbitrary point *P* holds:

$$\frac{\left(R_2+R_3\right)^2R_1^2}{a^4}+\frac{\left(R_3+R_1\right)^2R_2^2}{b^4}+\frac{\left(R_1+R_2\right)^2R_3^2}{c^4}\ge\frac{4}{3} \qquad (39)$$

Conjecture 3: For *ΔABC* and arbitrary point *P* holds:

$$\frac{r_2^2+r_3^2}{\sin^2 A}+\frac{r_3^2+r_1^2}{\sin^2 B}+\frac{r_1^2+r_2^2}{\sin^2 C}\ge\frac{2}{9}\left(R_1+R_2+R_3\right)^2 \qquad (40)$$

Considering the exponent generalization of Corollary 12, we pose the conjecture:

Conjecture 4: Let *P* be an arbitrary point which does not coincide with the vertices of *ΔABC*. If $k\ge 2$, then

$$\frac{r_2^2+r_3^2}{R_1^2\cos^k\frac{A}{2}}+\frac{r_3^2+r_1^2}{R_2^2\cos^k\frac{B}{2}}+\frac{r_1^2+r_2^2}{R_3^2\cos^k\frac{C}{2}}\ge\frac{2^{k-1}}{3^{\frac{k}{2}-1}} \qquad (41)$$

REFERENCES

[1] Klamkin, M.S., *Math. Mag.*, **48**, 44, 1975.
[2] Bennett, G., *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, **577-598**, 39, 1977.
[3] Ma, T.Y., Hu, X., *Journal of Normal Colleges*, **6**(2), 18, 2001.
[4] Mitrinovic, D., Pecaric, J.E., Volenec, V., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publisher, Dordrecht, Boston, London, 1989.
[5] Liu, J., Several new inequalities for the triangle, Mathematics Competition, *Hunan Education Press Hunan, P.R.C.*, **15**, 80, 1992.
[6] Liu, J., *J. Inequal. Pure Appl. Math.*, **9**(2), art. 58, 2008.
[7] Liu, J., *Zhong Shuxue*, **5**, 16, 1996.
[8] Hayashi, T., *Tohoku Math J.*, **4**, 68, 1913.
[9] Liu, J., *Hunan Shuxue Tongxun*, **5**, 1998.

Manuscript received: 09.12.2010
Accepted paper: 06.01.2011
Published online: 01.02.2011