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ESTIMATES FOR $e^x - \left(1+\frac{x}{t}\right)^t$ AND APPLICATIONS

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Abstract. *The aim of this paper is to give a short proof of an inequality bounding $e^x - \left(1+\frac{x}{t}\right)^t$, stated in [1]. Also some improvements are given.*

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INTRODUCTION

The main result of [1], established via harmonic, logarithmic and arithmetic mean inequality, is the following double inequality

$$\frac{x^2e^x}{2t+x+\max\{x,x^2\}} < e^x - \left(1+\frac{x}{t}\right)^t < \frac{x^2e^x}{2t+x}, \qquad \left(x>0,\ t>\frac{1-x}{2}\right) \tag{1}$$

together with a dual inequality. See [1, Theorem 1].

We give a simple proof of (1), using the following elementary consequence of Lagrange Theorem:

$$x(a-b)b^{x-1} < a^x - b^x < x(a-b)a^{x-1}, \qquad (x>a,\quad 0<b<a). \tag{2}\#$$

It is considered in [1] $x=1$, $t=n\in N$ case to rediscover the inequality

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1}, \tag{3}$$

stated in [2, Problem 170]. However, inequality (3) is true even if n is any positive real number, and this fact motivated us to give a short proof of (1), starting from (3), with $n=t/x$:

$$\frac{ex}{2t+2x} < e - \left(1+\frac{x}{t}\right)^{\frac{t}{x}} < \frac{ex}{2t+x}. \tag{4}$$

Theorem 1. For all real numbers $x,\ t>0$, it holds:

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} < e^x - \left(1+\frac{x}{t}\right)^t < x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1} \tag{5}$$

The proof follows by (2), with $a=e$ and $b=\left(1+\frac{x}{t}\right)^{\frac{t}{x}}$.

Note that the upper bound in (5) is better than the upper bound in (1), since using (4), we obtain

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1} < x\frac{ex}{2t+x}e^{x-1} = \frac{x^2e^x}{2t+x}.$$

Numerical computations prove also the superiority of lower bound in (5) over lower bound (1). We are able to give a rigorous proof of this fact in $x \le 1$ case, when $\max\{x, x^2\} = x$. Indeed, using again (4), we deduce

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} > \frac{ex^2}{2t+2x}\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} = \left(\frac{1}{e}\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1}\frac{x^2e^x}{2t+2x} > \frac{x^2e^x}{2t+2x}.$$

In conclusion, being derived from (5), inequality (1) is a simple consequence of Lagrange Theorem. Notice also that (5) is true for all $x,\ t > 0$ a supplementary condition as required in (1) is not necessary.

Finally, we give

Theorem 2. The following inequality holds

$$e^x - \left(1+\frac{x}{t}\right)^t < \frac{(3t-2x)x^2e^x}{6t^2}. \tag{6}$$

whenever $x \ge 1$ and $t > \max\left\{\frac{2}{3}x+2,\ \frac{1}{4}x^2\frac{1}{12}\sqrt{9x^4-48x^3}\right\}$. If $1 \le x < \frac{16}{3}$, then it suffices $t > \frac{2}{3}x+2.$

Proof. First remark that (6) is a new improvement of upper bound (1), since

$$\frac{(3t-2x)x^2e^x}{6t^2} = \frac{x^2e^x}{2t+x} - \frac{(t+2x)x^3e^x}{6t^2(2t+x)} < \frac{x^2e^x}{2t+x}.$$

In order to prove inequality (6), it suffices to show that $f_t < 0,$ where

$$f_t(x) = \ln\left(1-\frac{(3t-2x)x^2}{6t^2}\right) + x - t\ln\left(1+\frac{x}{t}\right).$$

But f_t is strictly decreasing in x, since

$$f'_t(x) = -\frac{x^3(3t-2x-6)}{(t+x)(6t^2-3tx^2+2x^3)} < 0, \text{ with}$$

$$f_t(1) = \ln\left(\frac{1}{3t^2}-\frac{1}{2t}+1\right) - t\ln\left(\frac{1}{t}+1\right)+1 < 0, \qquad t > \frac{2}{3}+3.$$

Now $f_t(x) \le f_t(1) < 0$ for $x \ge 1$, which completes the proof.

By taking $t = n \ge 3 > \frac{8}{3}$ and $x = 1,$ we obtain the following refinement of upper bound (3)

$$e - \left(1+\frac{1}{n}\right)^n < \frac{(3n-2)e}{6n^2} = \frac{e}{2n+1} - \frac{(n+2)e}{6n^2(2n+1)} < \frac{e}{2n+1},$$

but we are convinced that eventually different methods can provide much better estimates (3).

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