ORIGINAL PAPER

EXPONENTIAL SERIES AND COMBINATORIAL PROBLEMS

VASILE POP

Technical University of Cluj-Napoca, Faculty of Automation and Computer Science, 400114, Cluj-Napoca, Romania

Abstract. *In this paper we analyze two problems of combinatorial geometry, colouring problems, that, by the solving method have a tight connection with the series* $e=\sum_{n=0}^{\infty}\frac{1}{n!}$. *Particular cases of these problems have been proposed at the International Mathematical Olympiads, editions V and XX.*

Keywords: *combinatorial geometry, colouring problems.*

1. INTRODUCTION

Surprisingly, some colouring and numbering problems (combinatorial geometry), which in particular forms have been given at International Olympiads, are connected with the series

$$e=\sum_{n=0}^{\infty}\frac{1}{n!} \quad \text{and} \quad e^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}.$$

The connection between the problems that we will present and the number e is based on the recurrence relationships satisfied by the partial sums sequences

$$e_n=1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}, \quad n\in\mathbb{N}$$

and by the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ defined by

$$a_n=n!e_n \quad \text{and} \quad b_n=1+a_n=1+n!e_n, \quad n\in\mathbb{N}.$$

(R1): $e_n=e_{n-1}+\frac{1}{n!},\ n\geq 1,\ e_0=1$

(R2): $a_n=na_{n-1}+1,\ n\geq 1,\ a_0=1$

(R3): $b_n=1+(n(b_{n-1}-1)+1),\ n\geq 1,\ b_0=2$.

Remark 1.1. The sequence $(a_n)_{n\in\mathbb{N}}$ can be defined by its general term:

$$a_n=[n!\cdot e], \quad n\geq 1.$$

We have:

$$n!\cdot e=n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots\right)$$
$$=n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)+n!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots\right)$$
$$=a_n+\varepsilon_n,$$

where

$$\varepsilon_n=\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$
$$=\frac{1}{n+1}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\ldots\right)$$
$$<\frac{1}{n+1}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\ldots\right)$$

$$= \frac{1}{n+1}\sum_{k=0}^{\infty}\frac{1}{(n+2)^k} = \frac{1}{n+1}\cdot\frac{1}{1-\dfrac{1}{n+2}} = \frac{n+2}{(n+1)^2} < 1, \text{ for } n \geq 1.$$

Since $a_n \in \mathbb{N}$ and $\varepsilon_n \in (0,1)$, we have: $[n!\cdot e] = a_n$ and $\{n!e\} = \varepsilon_n$ (we denoted $[x]$ the integer part of the number x and $\{x\}$ the fractionary part of x).

2. COLOURINGS WITH MONOCHROMATIC TRIANGLES

At the V-th O.I.M. edition in Moscow, the following problem was given:

Consider 17 points in the plane, each 3 of them non-collinear. Each segment that joins two points is coloured in one of the colours: red, yellow and blue. P rove that at least one monochromatic triangle is form ed (having the sides of the same colour). Following the argument made in solving the problem, this can be generalized like this:

Problem 2.1. For each natural number $n \geq 1$, consider a set B_n of

$$b_n = 1 + n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right)$$

points in plane, each three non-collinear. The segments that join two points are colo ured each with one of the n given colours. Prove that in any colouring there exist a m onochromatic triangles.

Solution. The sequence (b_n) satisfies the recurrence relation (R2) and $b_1 = 3$, relation that we use to prove our statement, by mathematical induction, with respect to $n \in \mathbb{N}^*$. For $n = 1$ we have in plane $b_1 = 3$ points (a triangle) and we colour the sides with one colour (the triangle is monochromatic). Assume the statement true for $n-1$ and prove it for n . We fix one of the given points $A_1 \in B_n$. It is joined with the other $b_n - 1$ points by segments of n colours.

Since

$$\frac{b_n - 1}{n} = \frac{n(b_{n-1}-1)+1}{n} > b_{n-1} - 1,$$

it follows that from the $b_n - 1$ segments, at least b_{n-1} have the same colour C_n . We have to analyze two situations:

1. If one of the segm ents that join two of the b_{n-1} points, for exam ple $[A_2A_3]$ has the colour C_n then the triangle $A_1A_2A_3$ has the sides of the same colour C_n .
2. If none of these segm ents is of colour C_n , we keep only these b_{n-1} points, which for m a set B_{n-1} in whi ch each seg ment that joins two poi nts is colou red with on e of the colours $C_1, C_2, \ldots, C_{n-1}$ and so the problem is reduced to the induction hypothesis. □

One can notice th at the sequence (b_n)_n increases very fast and that it is dete rmined in the problem only due to the argum ent (that we m ade).In a natural way, the following equivalent problems appear:

Open problem 2.2. Is the num ber b_n the sm allest number with the prope rty that in each n -colours colouring of the segm ents that join the points of the set B_n with $|B_n| = b_n$, at least a monochromatic triangle is formed?

Open problem 2.3. Which is the maxim um number of poi nts x_n that can be considered in plane (each three non-collin ear) for which there ex ists a colouring of the segm ents between them, with n colours, such that no monochromatic triangle is formed?

We believe that the problem s 2.2 and 2.3 will s tay (as m any problems of combinatorial geometry) open for a long time.

From the general case we have the evaluations

$$2^n \le x_n \le a_n, \quad n \in \mathbb{N}.$$

The first inequality is obtained by constructing a graph with 2^n vertices, inductively, which does not contain one -colour triangles. For $n = 1$ we take 2 point that are joined by a segment of colour C_1 . F or n we con sider two groups, each of 2^{n-1} points, joined in each group by segments of the colours $C_1, C_2, \dots, C_{n-1}$ such that no monochr omatic triangles are formed. The segments that join the vertices from two graphs are all co loured with the colour C_n . The second inequality follows from problem 2.1, since $x_n < b_n = a_n + 1$, so $x_n \le a_n$.

For the case $n = 1$ it is obvious that $x_1 = a_1 = 2$.

For the case $n = 2$ we have $x_2 = a_2 = 5$ (the sides of the pentagon are coloured with the colour C_1 and the diagonals with colour C_2).

For the case $n = 3$ we will s how, using an extr emely ingenious nonelementary and difficult construction that $x_3 = a_3 = 16 = b_3 - 1$.

More precisely, we will show that the di agonals and the sides of a regular polygon with 16 vertices can be coloured with three colours such that no monochromatic triangle is formed. Consider a set $V = \{v_1, v_2, v_3, v_4, v_5\}$ with 5 elem ents (the vertices of a regula r pentagon) and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_2 \cup \mathcal{P}_4$ the set of subsets of V which contain 0, 2 or 4 elements. We have $\mathcal{P}_0 = \emptyset$, $\mathcal{P}_2 = \mathcal{L} \cup \mathcal{D}$, where $\mathcal{L} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_1\}\}$ (the sides)

$\mathcal{D} = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_4, v_1\}, \{v_5, v_2\}\}$ (the diago nals). The set $\mathcal{P}$ has $1 + 5 + 5 + 5$ elements, and the pair $(\mathcal{P}, \Delta)$ is a subgroup of the group $(\mathcal{P}(V), \Delta)$, where the operation "Δ" is th e symmetric difference $A\Delta B = (A \cup B) \setminus (A \cap B)$. T he essential properties (for us) of the group $(\mathcal{P}, \Delta)$ are:

1. $X\Delta X = \emptyset,\ X \in \mathcal{P}$
2. $X \ne Y \Rightarrow X\Delta Y \ne \emptyset,\ X, Y \in \mathcal{P}$
3. $(X\Delta Y)\Delta(Y\Delta Z) = X\Delta Z,\ X, Y, Z \in \mathcal{P}$

from which follow:

$1'$. $X, Y \in \mathcal{L} \Rightarrow X\Delta Y \notin \mathcal{L}$

$2'$. $X, Y \in \mathcal{D} \Rightarrow X\Delta Y \notin \mathcal{D}$

$3'$. $X, Y \in \mathcal{P}_4 \Rightarrow X\Delta Y \notin \mathcal{P}_4$.

We consider now a bijective function from the set of the vertices of the polygon with 16 vertices to $\mathcal{P}$, $f : \{A_1, A_2, \dots, A_{16}\} \to \mathcal{P}$ and we colour each segm ent $[A_iA_j]$, $1 \le i < j \le 16$ with the colour $C[A_i, A_j] \in \{C_1, C_2, C_3\}$ like this:

$$C[A_iA_j] = C_1 \quad \text{if} \quad f(A_i)\Delta f(A_j) \in \mathcal{L}$$
$$C[A_iA_j] = C_2 \quad \text{if} \quad f(A_i)\Delta f(A_j) \in \mathcal{D}$$
$$C[A_iA_j] = C_3 \quad \text{if} \quad f(A_i)\Delta f(A_j) \in \mathcal{P}_4$$

For an arbitrary triangle $A_iA_jA_k$, if $C[A_iA_j] = C[A_jA_k] = C$ then from the relations 1, 2, 3, $1'$, $2'$, $3'$ it follows $C[A_iA_k] \ne C$ so the triangle $A_iA_jA_k$ is not a m onochromatic triangle.

Remark 2.4. The problem s solved in paragraph 2 are known in combinatorics and graph theory as Ramsey type problems. If $a_1, a_2, \dots, a_n$ are positive integers and $n \ge 2$ then there exists minimal positive integer $N = R_n(a_1, a_2, \dots, a_n)$ with the property such that for every colouring with n colours $C_1, C_2, \dots, C_n$ of the edges of com plete graph K_N , there exists $i \in \{1, 2, \dots, N\}$ and a complete subgraph with a_i vertices and all edges of C_i colour. $R_n(a_1, a_2, \dots, a_n)$ is called Ramsey number of parameters $a_1, a_2, \dots, a_n$. In a problem given to OIM - Moscow 1976, the assertion is $R_3(3, 3, 3) \le 17$. The problem 2.1 gives the next evaluation $R_n(3, 3, \dots, 3) \le b_n$, and in the last problem it was proved that $R_3(3, 3, 3) = 17 = b_3$. In colouring problem s, with two colours, one denotes with $R(m, n)$

the minimum number of vertices of a com plete graph with the edg es coloured with two colours C_1 and C_2, and for every colouring there exists a complete subgraph with m vertices and all edges of colour C_1 or a complete subgraph with n vertices and all edges of colour C_2.

3. THE EQUATION $x+y=z$ ON PARTITIONS OF A SET

At the XX-th O.I.M. edition, Bucharest 1978, the following problem was given [2, 5]:

At a congress are participating 1978 m athematicians from 6 countries, registered on a list from 1 to 1978. Prove that there exist three of the order numbers $x,y,z \in \{1,2,\dots,1978\}$ of some mathematicians from the same country, such that $x+y=z$.

Following the reasoning m ade in the solution of this problem, we can for mulate the following generalization:

Problem 3.1. The natural numbers from 1 to

$$a_n = n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}\right)$$

are partitioned in n sets. Show that the equation $x+y=z$ has a solution in at least one of the sets of the partition.

Solution. From the rec urrency relationship (R3), since $\frac{a_n}{n} > a_{n-1}$, it follows th at at leas t $a_{n-1}+1$ num bers are in the same set A_1 of the partition. Let them be $k_1 < k_2 < \dots < k_{a_{n-1}+1}$. W e consider the differences $k_2-k_1 < k_3-k_1 < \dots < k_{a_{n-1}+1}-k_1$. If one of these num bers, for exam ple $k_i - k_1$, is form A_1 then we denote $x=k_1$, $y=k_i-k_1$, $z=k_i$ and $x+y=z$ with $x,y,z \in A_1$. If the a_{n-1} differences are from $A_2 \cup \dots \cup A_n$ then at least $a_{n-2}+11$ are from the same set A_2.

Let them be $u_1 < u_2 < \dots < u_{a_{n-2}+1}$ and we consider the differences $u_2-u_1 < u_3-u_1 < \dots < u_{a_{n-2}+1}-u_1$. W e notice that the d ifferences are of the for m $u_i - u_1 = (k_p-k_1)-(k_q-k_1) = k_p-k_q$. If one of these is in A_1 we have a solution of the equation $x+y=z$ in A_1 , and if one of the differences is in A_2 we have a solution in the set A_2 . If not, we remain with a_{n-2} differences distributed in $(n-2)$ sets and we continue the induction.

Remark. For $n=1,2,3,4,5$ we get $a_1=2$, $a_2=5$, $a_3=16$, $a_4=65$ and $a_5=1957<1978$.

We can formulate problems similar to 2.2 and 2.3.

Open problem 3.2. Which is the m inimum number y_n such that any partition of the set $\{1,2,\dots,y_n\}$ in n sets has the prope rty that the equation $x+y=z$ has a solu tion in one of the sets of the partition?

Open problem 3.3. Which is the maximum number z_n with the property that the se t $\{1,2,\dots,z_n\}$ can be partitioned in n sets $A_1 \cup \dots \cup A_n$ such that, if $x,y \in A_i$, then $x+y \notin A_i$, $i=\overline{1,n}$?

It can be proved that $y_1=a_1$, $y_2=a_2$, $y_3=a_3$ and $z_1=a_1-1$, $z_2=a_2-1$, $z_3=a_3-1$, and for the general case, the problems remain open.

4. THE SOLVING OF THE EQUATION $x+y=z$ WITH MONOCHROMATIC TRIANGLES

Although problems 2.1 and 3.1 seem essentially different, the connection between the numbers a_n and b n, $b_n = 1 + a_n$ suggests that between the two of them might exist a deeper connection, which in fact shows that the two problems 2.1 and 3.1 and equivalent.

Definition 4.1. We say that the natural number β_n has the property Q_n if for each colouring of the sides of a complete graph with β_n vertices, using segments of n colours, at least a one monochromatic triangle is formed.

Definition 4.2. We say that the natural number α_n has the property P_n if for every partition of the set of natural numbers $\{1,2,\ldots,\alpha_n\}$ in n sets, there exist the numbers x, y, z in the same set such that $x+y=z$.

Theorem 4.3. *The number α_n has the property P_n if and only if the number $\beta_n = \alpha_n + 1$ has the property Q_n .*

Proof. If α_n has the property P_n then we consider the graph with vertices $A_0, A_1, A_2, \ldots, A_{\alpha_n}$ where A_0 is the center of half a circle and the points $A_1, A_2, \ldots, A_n$ are situated on the half-circle; graph with $\beta_n = \alpha_n + 1$ vertices. If α_n has the property P_n and $M_1 \cup M_2 \cup \ldots \cup M_n$ is an arbitrary partition of the set $\{1,2,\ldots,\alpha_n\}$ then we colour the segment $[A_iA_j]$ with the colour C_p , where p is defined by $|i-j| \in M_p$, $p=\overline{1,n}$.

Let us notice that the triangle $A_iA_jA_k$ is monochromatic if and only if the numbers $j-i$, $k-j$ and $k-i$ $(i \le j \le k)$ are in the same set M_p of the partition. Denoting $x=j-i$, $y=k-i$ and $z=k-i$ we get $x+y=z$ (solution of the equation in the set M_p).

Remark 4.4. Between the numbers x_n from the open problem 2.3 and y_n from the open problem 3.2, there exists the relation $x_n = y_n$. The previous numbers are called Schur numbers (for the relation with Ramsey numbers sees [1] and [4]). Since for every partition of the set of positive integers in a finite number of subsets there exist a subset in which the equation $x+y=z$ has a solution, we say that $x+y=z$ is a normal equation (I. Schur).

5. PERMUTATION WITHOUT FIXED POINTS

An example of application of inclusion and exclusion principle is the problem of finding the number of permutations of the set \{1,2,\ldots ,n\} without fixed points. The result obtained is related to the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}.$$

Denoting by A_i the set of permutations having i as a fixed point, $i=\overline{1,n}$, then the set of all permutations having at least a fixed point is $\bigcup_{i=1}^{n} A_i$. We have:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \ldots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

and

$$|A_i| = (n-1)!,\ \ |A_i \cap A_j| = (n-2)!, \ldots, \left|\bigcap_{i=1}^{n} A_i\right| = 1$$

Subtracting from the number of all permutations the number of permutations with at least a fixed point we get the number d_n of all permutations without fixed points:

$$d_n = n! - (n(n-1)! - C_n^2(n-2)! + C_n^3(n-3)! - \ldots + (-1)^{n-1})$$
$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

Remark that the probability of choosing a permutation without fixed points is

$$p_n = \frac{d_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}$$

and $\lim_{n\to\infty} p_n = e^{-1}$.

Remark. For the sequence $(d_n)_{n\geq 1}$ the following relations hold:
1. $d_{n+1} = (n+1)d_n + (-1)^{n+1}$
2. $d_{n+1} = n(d_n + d_{n-1})$.

REFERENCES

[1] Chung, F.R.K., *Discrete Math.*, **5**, 317, 1973.
[2] Cuculescu, I., *Olimpiadele internationale de matematica ale elevilor*, Ed. Tehnica, Bucuresti, 1984.
[3] Exoo, G., *SIAM J. Discrete Math.*, **2**, 488, 1989.
[4] Fridricksen, H. J., *Combin. Theory, Ser. A*, **27**, 376, 1979.
[5] Morozova, E.A., Petrakov, P.S., Skvotov, V.A., *Olimpiadele internationale de matematica*, Ed. Tehnica, Bucuresti, 1978.
[6] Tomescu, I., *Introducere in combinatorica*, Ed. Tehnica, Bucuresti, 1972.
[7] Tomescu, I., *Probleme de combinatorica si teoria grafurilor*, Ed. Didactica si Pedagogica, Bucuresti, 1981.

Manuscript received: 15.10.2010
Accepted paper: 02.12.2010
Published online: 01.02.2011