ORIGINAL PAPER

ON A CONJECTURE BY BENCZE

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Abstract. *The aim of this paper is to discuss an open problem posed by Bencze in [Open question 3108, Octogon Math. Mag, vol. 16, no. 2b, October 2008, page 1237] about Euler's number and some means.*

Keywords: *Euler's number, arithmetic mean, generalized arithmetic mean, geometric mean, logarithmic mean, inequalities, approximations.*

MSC: *11Y60, 40A05, 33B15, 26D15.*

INTRODUCTION

Motivated by the following representation of Euler's number

$$e=\left(1+\frac{1}{n}\right)^{L(n,n+1)}$$

where

$$L(x,y)=\frac{x-y}{\ln x-\ln y},\ \ (x,y>0)$$

is the logarithmic mean, Bencze [1] proposed the following interesting conjecture

$$e<\left(1+\frac{1}{n}\right)^{\left(\frac{n^{\frac{1}{3}}+(n+1)^{\frac{1}{3}}}{2}\right)^3} \tag{1}$$

This can be written in terms of means as

$$e<\left(1+\frac{1}{n}\right)^{M_{\frac{1}{3}}(n,n+1)}$$

where for $t>0$,

$$M_t(x,y)=\left(\frac{x_t+y_t}{2}\right)^{\frac{1}{t}}\ \ (x,y>0)$$

is the generalized arithmetic mean.

Bencze's inequality (1) has attracted the interest through authors. In particular, Khattri and Witkowski [2] remarked that

$$\left(1+\frac{1}{n}\right)^{n}<\left(1+\frac{1}{n}\right)^{G(n+1,n)}<e<\left(1+\frac{1}{n}\right)^{A(n+1,n)}<\left(1+\frac{1}{n}\right)^{n+1} \quad (2)$$

where

$$G(x,y)=\sqrt{xy}\,,\quad A(x,y)=\frac{x+y}{2}\,,\quad (x,y>0)$$

and searched for the value t which provides the best approximation of the form

$$e\approx\left(1+\frac{1}{n}\right)^{M_t(n,n+1)}$$

The Result. We improve in this paper inequalities (1)-(2), giving the following

Theorem 1. For every integer $n\geq 4$, it holds

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}}<e<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}+\frac{3}{160n^4}}$$

The superiority of our new double inequal ity over (1)-(2) can be establish ed using computer softwares such as Maple which also says that

$$\lim_{n\to\infty}n^3\left(\left(\frac{n^{\frac{1}{3}}+(n+1)^{\frac{1}{3}}}{2}\right)^3-\left(n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}+\frac{3}{160n^4}\right)\right)=\frac{1}{6480}$$

and moreover

$$\left(\frac{n^{\frac{1}{3}}+(n+1)^{\frac{1}{3}}}{2}\right)^3=\left(n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}+\frac{3}{160n^4}\right)+\frac{1}{6480}+O\left(\frac{1}{n^4}\right).$$

Proof. We have to prove the double inequality

$$\frac{1}{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}+\frac{3}{160n^4}}<\ln\left(1+\frac{1}{n}\right)<\frac{1}{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}}$$

which follows from

$$\sum_{k=1}^{8} \frac{(-1)^{k+1}}{kn^k} < \ln\left(1+\frac{1}{n}\right) < \sum_{k=1}^{7} \frac{(-1)^{k+1}}{kn^k}$$

arising from Taylor's expansion of the logarithm function. Indeed,

$$\frac{1}{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}+\frac{3}{160n^4}} < \sum_{k=1}^{8} \frac{(-1)^{k+1}}{kn^k}$$

and

$$\sum_{k=1}^{7} \frac{(-1)^{k+1}}{kn^k} < \frac{1}{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}}$$

since

$$\frac{1}{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}+\frac{3}{160n^4}} - \sum_{k=1}^{8} \frac{(-1)^{k+1}}{kn^k} = -\frac{P(n)}{840n^8 Q(n)} < 0$$

and

$$\sum_{k=1}^{7} \frac{(-1)^{k+1}}{kn^k} - \frac{1}{n+\frac{1}{2}-\frac{1}{2n}+\frac{1}{24n^2}-\frac{19}{720n^3}} = -\frac{R(n)}{420n^7 S(n)} < 0$$

where

$$P(n) = 7230n - 14640n^2 + 29656n^3 - 110454n^4 - 22380n^5 + 17260n^6 - 2835$$

$$Q(n) = 60n^2 - 38n - 120n^3 + 720n^4 + 1440n^5 + 27$$

$$R(n) = 7296n^2 - 3130n - 30315n^3 - 7150n^4 + 5670n^5 + 1140$$

$$S(n) = 30n - 60n^2 + 360n^3 + 720n^4 - 19$$

Finally, $P(n), Q(n), R(n), S(n) > 0$ for every $n \geq 4$, since

$$P(n) = 21193445 + 50436414(n-4) + 41692848(n-4)^2 + \\ +16774392(n-4)^3 + 3584346(n-4)^4 + 391860(n-4)^5 + 17260(n-4)^6$$

$$Q(n) = 2089 + 9802(n-1) + 18420(n-1)^2 + 17160(n-1)^3 + 7920(n-1)^4 + 1440(n-1)^5$$

$$R(n) = 37569 + 746291(n-3) + 879261(n-3)^2 + 394185(n-3)^3 + 77900(n-3)^4 + 5670(n-3)^5$$
$$S(n) = 1031 + 3870(n-1) + 5340(n-1)^2 + 3240(n-1)^3 + 720(n-1)^4.$$

REFERENCES

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[2] Khattri, S.K., Witkowski, A., *RGMIA Research Report Collection*, **12** (4), Art. 12, 2009.

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