ORIGINAL PAPER

GENERALIZATION OF PTOLEMY'S THEOREMS

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Abstract. *In this paper we present the generalizations of the Ptolemy's theorems, and after then we present some interesting applications.*
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Theorem 1. (M. Bencze, 1984). If $A_1A_2...A_n$ is a convex polygon inscripted into a circle, then the following identity is valid:

$$\frac{A_2A_n}{A_1A_2 \cdot A_1A_n} = \frac{A_2A_3}{A_1A_2 \cdot A_1A_3} + \frac{A_3A_4}{A_1A_3 \cdot A_1A_4} + ... + \frac{A_{n-1}A_n}{A_1A_{n-1} \cdot A_1A_n}$$

Proof. Let $C\ (O,\ R)$ be the circle which we have the $A_1A_2...A_n$ convex polygon inscribed in. Applying the $T(A_1,\ t)$ ($t \neq 0$) inversion, the $C(O,\ R)$ circle, which we have the A_1 point picked out of, transform in a straight line (d) which is perpendicular to A_1O. With this inversion we get $B_k = T(A_k)$ (k = 2, 3, ..., n) points. These points $B_2, B_3, ..., B_n$ are on the (d) line in order of their indexes. According to the property of inversion we have the following equalities:

$$B_2B_n = |t| \cdot \frac{A_2A_n}{A_1A_2 \cdot A_1A_n},\ B_2B_3 = |t| \cdot \frac{A_2A_3}{A_1A_2 \cdot A_1A_3},\ B_3B_4 = |t| \cdot \frac{A_3A_4}{A_1A_3 \cdot A_1A_4},\ ...,\ B_{n-1}B_n = |t| \cdot \frac{A_{n-1}A_n}{A_1A_{n-1} \cdot A_1A_n}$$

Knowing that $B_2B_n = B_2B_3 + B_3B_4 + ... + B_{n-1}B_n$ and substituting the relation written above, we got the result of the theorem.

Application 1.1. If $ABCD$ is a convex and concyclic quadrilateral, then

$$AC \cdot BD = AB \cdot CD + BC \cdot DA$$

Proof. In Theorem 1 we take n=4. This is the *classical* first Theorem of Ptolemy. Therefore Theorem 1 is a generalization of Ptolemy's first Theorem.

Application 1.2. If ABCDEF is a convex and concyclic hexagon, then

$$AD \cdot BE \cdot CF = AB \cdot ED \cdot CF + BC \cdot EF \cdot AD + CD \cdot FA \cdot BE + AB \cdot CD \cdot EF + BC \cdot DE \cdot FA$$

Proof. In theorem 1 we take $n=6$ etc.

Application 1.3. Let $A_1A_2...A_n$ be a regular polygon inscripted into a circle. On this circle we take the point $M \in \overset{\frown}{A_1A_2}$, then we have the following identity:

$$\frac{1}{MA_1 \cdot MA_2} = \frac{1}{MA_2 \cdot MA_3} + \frac{1}{MA_3 \cdot MA_4} + \ldots + \frac{1}{MA_n \cdot MA_1}$$

Proof. We use the Theorem 1 for the concyclic polygon $B_1B_2...B_nB_{n+1}$, where $B_1=M$, $B_k=A_k$ ($k = 1, 2, ..., n$) and $B_{n+1}=A_1$.

Application 1.4. If $z_k \in \mathbb{C}$ $(k=1, 2, ..., n)$ $z_i \neq z_j$ $(i \neq j)$, then $|z_1| = |z_2| = ... = |z_n|$ if and only if

$$\frac{|z_n - z_2|}{|z_2 - z_1| \cdot |z_n - z_1|} = \frac{|z_3 - z_2|}{|z_2 - z_1| \cdot |z_3 - z_1|} + \frac{|z_4 - z_3|}{|z_3 - z_1| \cdot |z_4 - z_1|} + \ldots + \frac{|z_n - z_{n-1}|}{|z_{n-1} - z_1| \cdot |z_n - z_1|}$$

Proof. Let be $A_k(z_k)$ ($k = 1, 2, ..., n$), then $A_iA_j = |z_j - z_i| \left(i, j \in \{1, 2, ..., n\}, i \neq j\right)$ and the result follows from Theorem 1.

Application 1.5. If $x_k \in \mathbb{R}$ $(k=1, 2, ..., n)$ and $x_i - x_j \notin \{k\pi / k \in \mathbb{Z}\}$, then

$$\frac{|\sin(x_n - x_2)|}{|\sin(x_2 - x_1)\sin(x_n - x_1)|} = \frac{|\sin(x_3 - x_2)|}{|\sin(x_2 - x_1)\sin(x_3 - x_1)|} + \frac{|\sin(x_4 - x_3)|}{|\sin(x_3 - x_1)\sin(x_4 - x_1)|} + \ldots + \frac{|\sin(x_n - x_{n-1})|}{|\sin(x_{n-1} - x_1)\sin(x_n - x_1)|}$$

Proof. In Application 1.4 we take $z_k = \cos 2x_k + i \sin 2x_k$ $\left(k=1, 2, ..., n\right)$.

Theorem 2. (A generalization of Ptolemy's inequality). If $A_1A_2...A_n$ is a convex polygon, then

$$\frac{A_2A_n}{A_1A_2 \cdot A_1A_n} \leq \frac{A_2A_3}{A_1A_2 \cdot A_1A_3} + \frac{A_3A_4}{A_1A_3 \cdot A_1A_4} + \ldots + \frac{A_{n-1}A_n}{A_1A_{n-1} \cdot A_1A_n}$$

Proof. We suppose that the convex polygon $A_1A_2...A_n$ is not cyclic. Let C (O, R) be the circle which we have the triangle $A_1A_2A_n$ inscribed in. Applying the $T(A_1, t)$ $(t \neq 0)$ inversion the circle, which we have the A_1 point picked out, transform in a straight line *(d)* which is perpendicular to A_1O. With this inversion we get the $B_k = T(A_k)$ $(k=2, 3, ..., n)$ points. All of these points $B_2, B_3, ..., B_n$ are not on the *(d)* line, but $B_2, B_3, ..., B_n$ is a closen broken line, therefore

$$B_2B_n \leq B_2B_3 + B_3B_4 + \ldots + B_{n-1}B_n$$

and from this follows the results. See the proof of Theorem 1.

Application 2.1. If ABCD is a convex quadrilateral, then

$$AC \cdot BD \leq AB \cdot CD + BC \cdot DA.$$

This is the "classical" Ptolemy's inequality.

Application 2.2. If ABCDEF is a convex hexagon, then

$$AD \cdot BE \cdot CF \le AB \cdot ED \cdot CF + BC \cdot EF \cdot AD + CD \cdot FA \cdot BE + AB \cdot CD \cdot EF + BC \cdot DE \cdot FA$$

Application 2.3. If $z_k \in \mathbb{C}$ $(k = 1,\ 2,\ ...,\ n)$ $z_i \ne z_j$ $\left(i,\ j \in \{1,\ 2,\ ...,\ n\},\ i \ne j\right)$, then

$$\frac{|z_n - z_2|}{|z_2 - z_1| \cdot |z_n - z_1|} \le \frac{|z_3 - z_2|}{|z_2 - z_1| \cdot |z_3 - z_1|} + \frac{|z_4 - z_3|}{|z_3 - z_1| \cdot |z_4 - z_1|} + ... + \frac{|z_n - z_{n-1}|}{|z_{n-1} - z_1| \cdot |z_n - z_1|}$$

Theorem 3. (A generalization of second Theorem of Ptolemy). If $A_1A_2...A_n$ is a concyclic and convex polygon, then

$$\frac{A_2A_n}{A_1A_2 \cdot A_1A_n} \cdot \sum_{k=3}^{n-1} \frac{1}{A_1A_k^2} = \frac{1}{A_1A_2^2 \cdot A_1A_n} \cdot \sum_{k=3}^{n-1} \frac{A_kA_n}{A_1A_k} + \frac{1}{A_1A_2 \cdot A_1A_n^2} \cdot \sum_{k=3}^{n-1} \frac{A_2A_k}{A_1A_k} - \frac{A_2A_n}{A_1A_2^2 \cdot A_1A_n^2} \cdot \sum_{k=3}^{n-1} \frac{A_2A_k \cdot A_kA_n}{A_1A_k^2}$$

Proof. We use the notations and the inversion from proof of Theorem 1.
In triangle $A_1B_2B_n$ for the point $B_k \in B_2B_n$ we apply the Stewart theorem, so we obtain:

$$A_1B_k^2 \cdot B_2B_n = A_1A_2^2 \cdot B_kB_n + A_1B_n^2 \cdot B_2B_k - B_2B_n \cdot B_2B_k \cdot B_kB_n,$$

but

$$A_1B_2 = \frac{|t|}{A_1A_2}$$

$$A_1B_k = \frac{|t|}{A_1A_k}$$

$$A_1B_n = \frac{|t|}{A_1A_n}$$

$$B_2B_k = |t| \frac{A_2A_k}{A_1A_2 \cdot A_1A_k}$$

$$B_2B_n = |t| \frac{A_2A_n}{A_1A_2 \cdot A_1A_n}$$

$$B_kB_n = |t| \frac{A_kA_n}{A_1A_k \cdot A_1A_n}$$

therefore

$$\frac{A_2A_n}{A_1A_2 \cdot A_1A_n} \cdot \frac{1}{A_1A_k^2} = \frac{1}{A_1A_2^2 \cdot A_1A_n} \cdot \frac{A_kA_n}{A_1A_k} + \frac{1}{A_1A_2 \cdot A_1A_n^2} \cdot \frac{A_2A_k}{A_1A_k} - \frac{A_2A_n}{A_1A_2^2 \cdot A_1A_n^2} \cdot \frac{A_2A_k \cdot A_kA_n}{A_1A_k^2}$$

for all $k \in \{3,\ 4,\ ...,\ n\}$ and finally after addition we obtain:

$$\frac{A_2A_n}{A_1A_2 \cdot A_1A_n} \cdot \sum_{k=3}^{n-1} \frac{1}{A_1A_k^2} = \frac{1}{A_1A_2^2 \cdot A_1A_n} \cdot \sum_{k=3}^{n-1} \frac{A_kA_n}{A_1A_k} + \frac{1}{A_1A_2 \cdot A_1A_n^2} \cdot \sum_{k=3}^{n-1} \frac{A_2A_k}{A_1A_k} - \frac{A_2A_n}{A_1A_2^2 \cdot A_1A_n^2} \cdot \sum_{k=3}^{n-1} \frac{A_2A_k \cdot A_kA_n}{A_1A_k^2}$$

Application 3.1. If ABCD is a convex and concyclic quadrilateral, then

$$\frac{AC}{BD}=\frac{AB\cdot AD+CB\cdot CD}{BA\cdot BC+DA\cdot DC}.$$

Proof. In Theorem 3 we take n=4. This is the *classical* second Theorem of Ptolemy.

Application 3.2. If $z_k\in\mathbb{C}\quad (k=1,\ 2,\ ...,\ n)\ \ z_i\neq z_j\ \ \left(i,\ j\in\{1,\ 2,\ ...,\ n\},\ i\neq j\right)$, then

$$|z_1|=|z_2|=...=|z_n|$$

if and only if

$$\frac{|z_n-z_2|}{|z_2-z_1|\cdot|z_n-z_1|}\sum_{k=3}^{n-1}\frac{1}{|z_k-z_1|^2}=$$

$$=\frac{1}{|z_2-z_1|^2\cdot|z_n-z_1|}\sum_{k=3}^{n-1}\frac{|z_n-z_k|}{|z_k-z_1|}+\frac{1}{|z_2-z_1|^2\cdot|z_n-z_1|}\sum_{k=3}^{n-1}\frac{|z_k-z_2|}{|z_k-z_1|}-\frac{|z_n-z_2|}{|z_2-z_1|^2\cdot|z_n-z_1|}\sum_{k=3}^{n-1}\frac{|z_k-z_2|\cdot|z_n-z_k|}{|z_k-z_1|^2}$$

Proof. See the proof of the Application 1.4.

Application 3.3. If $x_k\in\mathbb{R}\ (k=1,\ 2,\ ...,\ n)$ and $x_i-x_j\notin\{k\pi/k\in\mathbb{Z}\}$, then

$$\frac{|\sin(x_n-x_2)|}{|\sin(x_2-x_1)\sin(x_n-x_1)|}\sum_{k=3}^{n-1}\frac{1}{\sin^2(x_k-x_1)}=\frac{1}{\sin^2(x_2-x_1)|\sin(x_n-x_1)|}\sum_{k=3}^{n-1}\left|\frac{\sin(x_n-x_k)}{\sin^2(x_k-x_1)}\right|+$$

$$+\frac{1}{|\sin(x_2-x_1)|\sin^2(x_n-x_1)}\sum_{k=3}^{n-1}\left|\frac{\sin(x_k-x_2)}{\sin^2(x_k-x_1)}\right|-\frac{|\sin(x_n-x_2)|}{\sin^2(x_2-x_1)\sin^2(x_n-x_1)}\sum_{k=3}^{n-1}\frac{|\sin(x_k-x_2)\sin(x_n-x_k)|}{\sin^2(x_k-x_1)}$$

Proof. In Application 3.2. we take $z_k=\cos 2x_k+i\sin 2x_k\ \ \left(k=1,\ 2,\ ...,\ n\right)$.

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