ORIGINAL PAPER

FINITENESS PROPERTIES FOR THE PATH COALGEBRA ASSOCIATED TO A POSET

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Abstract. *Let P be a partially ordered set (poset) locally finite and kQ the path coalgebra over a field k associated to P. In this paper we investigate finiteness properties of this coalgebra by using an injective morphism of coalgebras from incidence coalgebra kS of P to the path coalgebra kQ. We deduce that kQ is left semiperfect only if kS have the same property, and that kQ is cosemisimple when the order relation on P is the equality. Finally we characterize the coradical filtration of the path coalgebra.*

Keywords: *finiteness properties, coalgebra, poset, injective morphism, coradical filtration.*

1. INTRODUCTION AND PRELIMINARIES

Let *k* be a field. The aim of this paper is to study finiteness conditions for a well-known class of coalgebras, namely for path coalgebras associated to locally finite partially ordered sets. Path coalgebras provide a good framework for interpreting several combinatorial problems in terms of coalgebras.

Let $(P,\leq)$ be a partially ordered set (poset for short) and locally finite, i.e. the interval $[x,y]=\{z\in P / x\leq z\leq y\}$ is finite for any $x\leq y$ from *P*.

In [6] is proved the fact that to any poset *P* we can associate an oriented quiver $Q=(Q_0,Q_1)$, where:

- $Q_0 = P$, i.e. the set of vertices of *Q* is *P*;
- and for any $x\leq y$ from *P* let $\alpha : x\to y$ be the arrow from *x* to *y*, and if *x* is not comparable with *y* in *P* we don't have any arrow from *x* to *y*; we denote with Q_1 the set of all these arrows between all the vertices from Q_0.

It is obvious that the quiver $Q=(Q_0,Q_1)$ has no oriented cycles.

Now we can construct a $k-$ vector space, *kQ*, of base *Q*. This vector space becomes coalgebra over *k* with the following two linear applications:

- comultiplication:

$$\Delta_Q : kQ\to kQ\otimes kQ$$

$$\Delta_Q(p)=\sum_{p=p_1p_2} p_1\otimes p_2 = s(p)\otimes p+\sum_{i=1}^{n-1}\alpha_1...\alpha_i\otimes\alpha_{i+1}...\alpha_n + p\otimes t(p)\ ,$$

where $p=\alpha_1...\alpha_n$ is a path in *Q*, and

$\Delta_Q(e_x)=x\otimes x$, where e_x is the trivial path from *x* to *x* in *Q*, $x\in Q_0$

- counity:

$$\varepsilon_Q : kQ \to k$$

$\varepsilon_Q(p) = 0$, where $p = \alpha_1 ... \alpha_n$ is a path in *Q*, and

$\varepsilon_Q(e_x) = 1$, where e_x is the trivial path from *x* to *x* in *Q*, $x \in Q_0$.

So we can identify the trivial path e_x from *x* to *x* with the vertex *x*.
Also, if $S = \{[x, y] / x, y \in P, x \le y\}$ be the set of all intervals from *P* and

$$kS = \left\{ \sum_{i=1}^{n} a_{[x_i, y_i]}[x_i, y_i] / [x_i, y_i] \in S, a_{[x_i, y_i]} \in k, n \in \mathbf{N} \right\}$$

the *k* vector space of base *S*, we remind that *kS* becomes a *k* – coalgebra, called the incidence coalgebra of *P*, with the following two linear applications:

$$\Delta_S : kS \to kS \otimes kS \ , \ \Delta_S([x, y]) = \sum_{x \le z \le y} [x, z] \otimes [z, y] \text{ and}$$

$$\varepsilon_S : kS \to k \ , \ \varepsilon_S([x, y]) = \delta_{x,y} ,$$

where by $\delta_{x,y}$ we denote Kronecker's delta.

We also recall that if *C* is a coalgebra, then *C* is a left *C*, right *C* - bicomodule with coactions defined by the comultiplication, and this makes *C* a left C^*, right C^* - bimodule. Denote by $c^* \cdot c$ and $c \cdot c^*$ the left and the right actions of $c^* \in C^*$ on $c \in C$.

In [6] it is prooved an important result, precisely a relation between these two coalgebras.

Theorem 1.1. Let $(P, \le)$ a poset locally finite, $Q = (Q_0, Q_1, s, t)$ the quiver associated to *P*, $(kQ, \Delta_Q, \varepsilon_Q)$ the path coalgebra and $(kS, \Delta_S, \varepsilon_S)$ the incidence coalgebra of *P*. Then there is an injective morphism of *k* – coalgebras, $f : kS \to kQ$.

This morphism associate to any interval $[x, y]$ from the base *S* of the incidence coalgebra *kS*, the sum of all paths from *Q* starting from *x* and arriving to *y*, i.e. $f([x, y]) = \sum_{\substack{p \text{ is a path} \\ \text{from } x \text{ to } y}} p$. It is obvious that the application f is an injective morphism of coalgebras.

Through this morphism we can consider that the coalgebra *kS* is a subcoalgebra of the path coalgebra *kQ*.

We consider some finiteness properties for coalgebras.

A coalgebra *C* is called right semiperfect if the category of right *C* - comodules has enough projectives.

Theorem 1.2. A coalgebra C is right semiperfect if the injective envelope of any left C – simple comodule is finite dimensional. Similarly, a coalgebra C is left semiperfect if the injective envelope of any right C – simple comodule is finite dimensional.

Proposition 1.3. If C is a right semiperfect coalgebra and A is subcoalgebra of C, then A is also right semiperfect.

A coalgebra C is right or left cosemisimple if $C = C_0$, where C_0 the coradical of C, i.e. the sum of all simple subcoalgebras of C.

In [3] we find the following results:

Theorem 1.4. If P is a poset locally finite, then the incidence coalgebra kS associated to P is left semiperfect if and only if for any element $x \in P$ the set $\{y \in P / x \le y\}$ is finite.

Observation 1.5. Similarly, coalgebra kS is right semiperfect if and only if the injective envelope of any left kS comodule is finite dimensional, means that for any $x \in P$ the set $\{y \in P / y \le x\}$ is finite.

Throughout this paper we work over a field k. For basic definitions and notations on coalgebras we refer to [1] and [5].

2. FINITENESS PROPERTIES FOR THE PATH COALGEBRA ASSOCIATED TO A POSET

Recall that for any oriented quiver $Q = (Q_0, Q_1)$, the path coalgebra kQ is the k – vector space generated by all paths from Q and the comultiplication and counit are:

$$\Delta(\alpha) = \sum_{\beta\gamma=\alpha} \beta \otimes \gamma$$

$$\varepsilon(\alpha) = \begin{cases} 0, \text{ if } |\alpha| > 0 \\ 1, \text{ if } |\alpha| = 0 \end{cases},$$

where $\beta\gamma$ is the concatenation of β and γ, and $|\alpha|$ is the length of the path α, and

$$\varepsilon_Q(p) = \begin{cases} 0, \text{if } \ p \text{ is a notrivial path} \\ 1, \text{ if } \ p \text{ is a vertex.} \end{cases}$$

In [4], for some vertex $s \in Q_0$ of the quiver Q is noted with $Q(s \to)$ and $Q(\to s)$ the set of all paths from Q which starts from s, respectively ends in s. We have the following result:

The path coalgebra *kQ* is left semiperfect if and only if for any vertex $s \in Q_0$, the set $Q(s \to)$ is finite. In the same way, the path coalgebra *kQ* is right semiperfect if and only if the set $Q(\to s)$ is finite.

Now, considering the path coalgebra associated to a locally finite poset we have:

Theorem 2.1. Let *P* be a poset locally finite. The following assertions are equivalent:

i) incidence coalgebra *kS* is left semiperfect;

ii) path coalgebra *kQ* is left semiperfect.

Proof. $i) \Rightarrow ii)$ From theorem 1.4 we have that incidence coalgebra *kS* of *P* is left semiperfect if for any $x \in P$ the set $\{y \in P / x \leq y\}$ is finite. Obviously results that the set $Q(x \to)$ of all paths starting from x is a finite one, so kQ is left semiperfect.

$ii) \Rightarrow i)$ From theorem 1 we have an injective morphism of k – coalgebras, $f : kS \to kQ$, such that the incidence coalgebra *kS* is a subcoalgebra of path coalgebra *kQ* . Because any subcoalgebra of a left semiperfect coalgebra is also left semiperfect, we have that *kS* is the same.

Lets now study when path coalgebra *kQ* is cosemisimple.

Theorem 2.2. Let *P* be a poset locally finite. The following assertions are equivalent:

i) incidence coalgebra *kS* is cosemisimple;

ii) path coalgebra *kQ* is cosemisimple.

iii) the order relation from *P* is equality.

Proof. In [2] it is proved $i) \Leftrightarrow iii)$.

When the order relation from *P* is equality the two coalgebras *kS* and *kQ* coincide, so $iii) \Rightarrow ii)$.

The implication $ii) \Rightarrow i)$ is true, because the coalgebra *kQ* being cosemisimple implies that *kS* is cosemisimple as a subcoalgebra of *kQ*.

3. CORADICAL FILTRATION FOR THE PATH COALGEBRA OF A POSET LOCALLY FINITE

The coradical C_0 have a very important role in the construction of a filtartion of a coalgebra *C*. We define recursively C_n such that: for anz pozitive integer $n \geq 1$, let

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C).$$

The properties of $\{C_n\}_{n \geq 1}$ are showed in the next theorem [3].

Theorem 3.1. For any pozitve integer $n \geq 0$, $\{C_n\}_{n\geq 1}$ is an ascendant chain of subcoalgebras of C such that:

1) $C_n \subseteq C_{n+1}$ and $C = \bigcup_{n\geq 0} C_n$;

2) $\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i}$.

Any chain of subspaces of *C* which satisfie the conditions 1) and 2) from the theorem above we call it the coradical filtartion of the coalgebra *C*.

If $x, y \in P$ and $x < y$, we saw that the interval $[x, y]$ has lenght *n* if for any sequence $x = u_0 < u_1 < ... < u_m = y$ we have $m \leq n$, and there exists such a sequence for $m = n$. We write $length([x, y]) = n$. We make a convention that $length([x, x]) = 0$ for any $x \in P$. Now we are able to describe the coradical filtartion of the inicdence coalgebra *kS* of *P*.

Proposition 3.2. The $(n+1)$ - term kS_n of the coradical filtration of *kS* is:

$$kS_n =< [x, y] / length([x, y]) \leq n > .$$

The *proof* is in [2].

Lets now find the coradical filtration for the path coalgebra *kQ* of a poset *P*.

Consider $(kQ, \Delta_Q, \varepsilon_Q)$ the path coalgebra of the locally finite poset *P*.

For any positive integer $n \in \mathrm{N}$ denote by kQ_n the *k* – vector subspace of *kQ* generated by the set $\{p \in Q / |p| \leq n\}$ of all paths from *Q* of maximum length *n*. In this way we obtain an ascendant chain of subcoalgebras $(kQ)_0 \subseteq (kQ)_1 \subseteq (kQ)_2 \subseteq ...$ of kQ.

Proposition 3.3. The chain $(kQ)_0 \subseteq (kQ)_1 \subseteq (kQ)_2 \subseteq ...$ is a coradical filtration for the path coalgebra *kQ* of the locally finite poset *P*.

Proof. We poove that the chain $(kQ)_0 \subseteq (kQ)_1 \subseteq (kQ)_2 \subseteq ...$ is a coradical filtration for *kQ*. So, for any path $p \in Q$ and any sub-paths p_1, p_2 of p we have $|p_1| + |p_2| \leq |p|$, so $\Delta_Q((kQ)_n) \subseteq \sum_{i=0}^{n} (kQ)_i \otimes (kQ)_{n-i}$. It is obvious that $kQ = \bigcup_{n\geq 0} kQ_n$ and the proof is done.

Corolarly 3.4. If *P* is a locally finite poset we have:

$$f(C_n) \subseteq kQ_n ,$$

where C_n, respectively kQ_n, is the n –th term of the coradical filtration of incidence coalgebra *kS*, respectively of path coalgebra *kQ* of *P*, and *f* is the morphism from the Theorem 1.1.

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