

GENERALIZED STATISTICS IN SYSTEMS OF INTERACTING PARTICLES

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Abstract. *After a very brief introduction to the concept of fractional exclusion statistics (FES), introduced by Haldane in Phys. Rev. Lett. 67, 937 (1991), I shall present some recent results regarding the basic properties of this statistics. These properties have been overlooked for almost two decades and because of that, the application of FES to concrete physical systems was, to a certain degree, arbitrary. Essentially, the FES is a generalization of the Pauli Exclusion Principle which allows one to describe systems of interacting particles (bosons or fermions) as systems of “ideal” particles, but which obey a more peculiar type of statistics. In the second part of the paper I shall present some application of FES by analyzing concrete systems of interacting particles.*

Keywords: *statistics, particles, FES, FQHE.*

1. INTRODUCTION

The concept of fractional exclusion statistics (FES) was introduced by Haldane in a very well-known paper, Ref. [1], and represents a generalization of the Pauli Exclusion Principle. The thermodynamics of the FES was deduced three years later, independently by several authors [2-4] and the topic have been intensively studied in general in connection to lower dimensional systems, like integrable one-dimensional (1D) quantum systems, fractional quantum Hall effect (FQHE), etc. Nevertheless, despite its wide applicability and the interest it triggered, some important and basic properties of the FES remained unknown for a long time (see [5-8]). Moreover, it appeared recently that FES is not only some exotic type of statistics, manifesting in low dimensional, more or less abstract models, but it is a necessary ingredient for the proper description of mesoscopic systems in general [7, 9, 10].

In Section 2 I shall present the basic formalism of FES whereas the new developments will be discussed in Section 3. In Section 4 I shall briefly present how the concept of FES can be applied to concrete systems of interacting quantum particles and the conclusions will be presented in Section 5.

2. BASICS OF FES

Haldane defined the fractional exclusion statistics as acting on Hilbert spaces of finite dimensions [1]. If we have only one such a space, in which we put N ideal bosons or fermions, then the number of microscopic configurations we have in the system is:

$$W_b = (G+N-1)!/[N!(G-1)!] \quad (\text{for bosons})$$

or

$$W_f = G!/[N!(G-N)!] \quad (\text{for fermions}).$$

Fractional exclusion statistics of parameter α is an interpolation between these two cases and the number of configurations is

$$W = \frac{[G + (N - 1)(1 - \alpha)]!}{\{N! [G - \alpha N - (1 - \alpha)]!\}}$$

we say that the addition of δN particles in the system reduces the number of available states in the system by $\alpha \delta N$ [1, 2].

Now let us generalize the problem to the case when we have more than one Hilbert space. The spaces are $H_0, H_1 \dots$ of dimensions $G_0, G_1 \dots$ and which contain $N_0, N_1 \dots$ particles. In this case we have the FES parameters α_{ij} , with $i, j=0, 1, \dots$. Mutual exclusion statistics is manifested between the spaces H_i and H_j ($i \neq j$) if $\alpha_{ij} \neq 0$ we say that the addition of δN_j particles in the space H_j changes the number of available states in the space H_i by $-\alpha_{ij} \delta N_j$. With these notations, the total number of configurations is [2]

$$W = \prod_i \frac{[G_i + N_i - 1 - \sum_j \alpha_{ij} (N_j - \delta_{ij})]!}{N_i! [G_i - 1 - \sum_j \alpha_{ij} (N_j - \delta_{ij})]!} \quad (1)$$

Having the number of microscopic configurations (1), if we assign the energy ϵ_i and the chemical potential μ_i to the states in the space i , we can calculate the grandcanonical partition function, Z [2],

$$Z = \sum_{\{N_i\}} W(\{N_i\}) \exp \left[\sum_i \beta N_i (\mu_i - \epsilon_i) \right] \quad (2)$$

and the total energy of the system in the given configuration $E = \sum_i N_i \epsilon_i$ we use the notation

$\beta = \frac{1}{k_B T}$ where T is the temperature of the system.

The most probable configuration, $\{N_i\}$, is obtained by maximizing Z with respect to the set $\{N_i\}$. If we introduce the notations $n_i = \frac{N_i}{G_i}$ and $\beta_{ij} = \frac{\alpha_{ij} G_j}{G_i}$ and assume that for each i both, G_i and N_i are sufficiently large, so that we can use the Stirling approximation $\left[\ln G_i! \approx G_i \ln \left(\frac{G_i}{e} \right) \text{ and } \ln N_i! \approx N_i \ln \left(\frac{N_i}{e} \right) \right]$, the maximization procedure gives us the system of equations,

$$n_i e^{\beta(\epsilon_i - \mu_i)} = \left[1 + \sum_k (\delta_{jk} - \beta_{jk}) n_k \right] \prod_j \left[\frac{1 - \sum_k \beta_{jk} n_k}{1 + \sum_k (\delta_{jk} - \beta_{jk}) n_k} \right]^{\alpha_{ij}} \quad (3)$$

The system (3) is solved easier if we denote $w_i \equiv n_i^{-1} - \sum_k \beta_{ik} n_k / n_i$. Using these notations, (3) becomes

$$(1 + w_i) \prod_j \left(\frac{w_j}{1 + w_j} \right)^{\alpha_{ji}} = e^{\beta(\epsilon_i - \mu_i)} \quad (4a)$$

and n_i 's can be calculated from another system,

$$\sum_j (\delta_{ij} + \beta_{ij}) n_j = 1 \quad (4b)$$

3. THE PROPERTIES OF THE FES PARAMETERS

The difficulties of Eqs. (4) become apparent if we observe that in physical systems the definitions of the species are, to a certain degree, arbitrary [5-7, 9]. If we change the division of our FES system into species, then eventually both sets of parameters, α_{ij} and β_{ij} , may change, modifying Eqs. (4) in a way which is not specified in the formalism.

To solve this problem we have determine the rules by which the FES parameters, α_{ij} , change at change of the particle species. For this let us consider a system made of only two species (for details see [6]), 0 and 1. We denote the exclusion statistics parameters of this system by α_{00} , α_{01} , α_{10} , and α_{11} , and we start in the standard way [1, 2], by writing the total number of configurations corresponding to N_0 particles of species 0 and N_1 particles of species 1 as:

$$W_{\{0,1\}} = \prod_i^{\{0,1\}} \frac{[G_i + N_i - 1 - \sum_j^{0,1} \alpha_{ij} (N_j - \delta_{ij})]!}{N_i! [G_i - 1 - \sum_j^{0,1} \alpha_{ij} (N_j - \delta_{ij})]!} \quad (5)$$

and the grandcanonical partition function of the system as

$$Z_{\{0,1\}} = W_{\{0,1\}} \prod_i^{\{0,1\}} e^{\beta N_i (\mu_i - \epsilon_i)} \quad (6)$$

Applying again the Stirling formula and maximizing $Z_{\{0,1\}}$, we end-up with the system

$$(1 + w_0) \left(\frac{w_0}{1 + w_0} \right)^{\alpha_{00}} \left(\frac{w_1}{1 + w_1} \right)^{\alpha_{10}} = e^{\beta(\epsilon_0 - \mu_0)} \quad (7a)$$

$$(1 + w_1) \left(\frac{w_0}{1 + w_0} \right)^{\alpha_{01}} \left(\frac{w_1}{1 + w_1} \right)^{\alpha_{11}} = e^{\beta(\epsilon_1 - \mu_1)} \quad (7b)$$

$$(w_0 + \alpha_{00}) N_0 + \alpha_{01} N_1 = G_0 \quad (7c)$$

$$\alpha_{10} N_0 + (w_1 + \alpha_{11}) N_1 = G_1 \quad (7d)$$

For large systems we can split any of the two species of particles into subspecies and obtain a thermodynamically equivalent system. So let us split for example species 1 into the subspecies 1₀ and 1₁, of dimensions G_{10} and G_{11} . In this way we describe the total system as consisting of the species 0, 1₀ and 1₁, of particle numbers N_0 , N_{10} and N_{11} , in Hilbert spaces of dimensions G_0 , G_{10} and G_{11} . Obviously,

$$N_{1_0} + N_{1_1} = N_1 \quad \text{and} \quad G_{1_0} + G_{1_1} = G_1 \quad (8)$$

Let us denote the exclusion statistics parameters of the *new* system by α'_{00} , α_{01_0} , α_{01_1} , $\alpha_{1_0 0}$, $\alpha_{1_0 1_0}$, $\alpha_{1_0 1_1}$, $\alpha_{1_1 0}$, $\alpha_{1_1 1_1}$. To obtain the consistency relations for the new exclusion statistics parameters, we use the consistency relations (8) and vary independently the particle numbers N_{1_0} , N_{1_1} and N_0 . In this way we obtain

$$\alpha'_{00} = \alpha_{00} \quad (9a)$$

$$\alpha_{01} = \alpha_{01_0} = \alpha_{01_1}, \quad (9b)$$

$$\alpha_{10} = \alpha_{1_0 0} + \alpha_{1_1 0}, \quad (9c)$$

$$\alpha_{11} = \alpha_{1_0 1_0} + \alpha_{1_1 1_0} = \alpha_{1_0 1_1} + \alpha_{1_1 1_1} \quad (9d)$$

We can extend these results to a system of arbitrary number of particle species. If we split any of the species, say species j , into a number of sub-species, j_0, j_1, \dots , then all the parameters α_{kl} , with both, k and l different from j , remain unchanged, whereas the rest of the parameters must satisfy the relations:

$$\alpha_{ij} = \alpha_{ij_0} = \alpha_{ij_1} = \dots, \text{ for any } i, i \neq j \quad (10a)$$

$$\alpha_{ji} = \alpha_{j_0 i} + \alpha_{j_1 i} + \dots, \text{ for any } i, i \neq j \quad (10b)$$

$$\begin{aligned} \alpha_{jj} &= \alpha_{j_0 j_0} + \alpha_{j_1 j_0} + \dots \\ &= \alpha_{j_0 j_1} + \alpha_{j_1 j_1} + \dots = \dots, \end{aligned} \quad (10c)$$

4. APPLICATIONS

In [7] I proposed the ansatz

$$\alpha_{ij} = \alpha_{ij}^{(e)} + \alpha_{ij}^{(s)} \delta_{ij} \quad (11)$$

for any i, j , which satisfy the Eqs (10). The parameters $\alpha_{ij}^{(e)}$ are *extensive* and satisfy the condition

$$\alpha_{ij}^{(e)} \equiv \alpha_{ij} G_i, \quad \forall i, j \quad (12)$$

whereas the parameters $\alpha_i^{(s)}$ always refer to only one species of particles and are not extensive.

Typically, in the literature we find exclusion statistics parameters of the (s) type [2, 11-17]. Therefore in general $\alpha_{ij} = 0$ for any $i \neq j$, so there is no mutual statistics in the system. In such a case, the thermodynamic calculations simplify considerably. Note also that the ideal Fermi gas corresponds to $\alpha_i^{(s)} = 1$ for any i .

4.1. FES IN A SYSTEM WITH FERMI LIQUID TYPE OF INTERACTION

I present here the model of [9], which is a generalization of the Murthy and Shankar model [11], widely used in FES [11-14, 16, 18, 19]. In this model the total energy of the system,

$$E = \sum_i \epsilon_i n_i + \frac{1}{2} \sum_{ij} V_{ij} n_i n_j \quad (13)$$

is split into *quasiparticle energies*, $E \equiv \sum_i \tilde{\epsilon}_i n_i$, with

$$\tilde{\epsilon}_i = \epsilon_i + \sum_{j=0}^{i-1} v_{ij} n_j + \frac{1}{2} v_{ii} n_i \quad (14)$$

In Eqs. (13) and (14) ϵ_i and n_i ($i=0, 1 \dots$) are the free-particle energy and population of the level i , respectively, whereas V_{ij} represents the interaction energy between a particle on the state i and a particle on the state j .

Going to the quasi-continuous limit, assuming that the single particle energy spectrum has the density of states $\sigma(\epsilon)$ and that the interaction energy depends only on the energies, we write Eq. (14) as

$$\tilde{\epsilon} = \epsilon + \int_0^\epsilon V(\epsilon, \epsilon') \sigma(\epsilon') n(\epsilon') d\epsilon' \quad (15)$$

In what follows I shall assume that the function $\tilde{\epsilon}(\epsilon)$ is bijective and therefore I shall use interchangeably both, $\tilde{\epsilon}(\epsilon)$ and $\epsilon(\tilde{\epsilon})$, whenever necessary.

The FES is manifested in this case along the quasiparticle energy axis due to the change of the density of states along this axis at the change of the particle density. The FES parameters were calculated in Refs. [7, 9] and it was obtained

$$\alpha_{\tilde{\epsilon}}^{(s)} = \frac{V(\epsilon, \epsilon) \sigma(\epsilon)}{1 + \int_0^\epsilon \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') d\epsilon'} \quad (16)$$

and

$$a_{\tilde{\epsilon}\tilde{\epsilon}_i} = \frac{V(\epsilon, \epsilon_i) + f(\tilde{\epsilon}, \tilde{\epsilon}_i)}{1 + \int_0^\epsilon \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') d\epsilon'} \left[\frac{d \log \sigma(\epsilon)}{d \epsilon} \right]_{\epsilon(\tilde{\epsilon})} \quad (17)$$

where

$$f(\tilde{\epsilon}, \tilde{\epsilon}_i) = \int_{\epsilon_i}^{\epsilon} \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') \left[\frac{\delta \epsilon'}{\delta \rho(\epsilon_i)} \right]_{\{\rho(\tilde{\epsilon})\}} d\epsilon' \quad (18)$$

In these equation, I denote by $\rho(\tilde{\epsilon}) \equiv \sigma(\tilde{\epsilon})n(\tilde{\epsilon})$ the particle density along the $\tilde{\epsilon}$ axis, whereas $\left[\frac{\delta \epsilon'}{\delta \rho(\epsilon_i)} \right]_{\{\rho(\tilde{\epsilon})\}}$ represents the functional derivative of ϵ' with respect to the particle density at energy $\tilde{\epsilon}_i$, when we keep fix $\tilde{\epsilon}(\epsilon')$.

In the simplified models of Refs. [11-14, 16, 18, 19], only the direct exclusion statistics parameters have been used and $a_{\tilde{\epsilon}\tilde{\epsilon}_i}$ was identically zero for any $\tilde{\epsilon}$ and $\tilde{\epsilon}_i$. We observe now from Eq. (17), that this happens whenever $\frac{d\sigma}{d\epsilon} = 0$.

4.2. FES IN 1D QUANTUM GAS IN THE THERMODYNAMIC BETHE ANSATZ

The 1D gas of quantum particles in the thermodynamic Bethe ansatz (TBA) have been analyzed before [3, 13, 20, 21] and are recognized in general as being a system which can be described by FES. Here I will show that its FES parameters are indeed of the type (11), clarifying in this way a confusion made in the literature between the intensive a_{ij} and extensive α_{ij} parameters.

We consider the typical gas of N spineless particles, bosons or fermions, on a ring of circumference L . We assume that the system is nondiffractive [22] and the asymptotic particle wave number, k , is determined by the equation [23]

$$Lk - \sum_{k'} \theta(k - k') = Lk^{(0)} \quad (19)$$

where $k(0) = 2\pi I(k)/L$ is the free-particle wavenumber, $I(k)$ is an integer that depends on k and $\theta(k - k')$ is the phase-shift due to the interaction.

To simplify the notations and to be also in accordance with Refs. [22, 23] we set the units so that $\hbar = m = 1$, where m is the mass of the particle. In these units the total number of particles, momentum and energy of the system are $N = \sum_k 1$, $P = \sum_k k$, and $E = \sum_k k^2 / 2$.

For large systems we transform the summations into integrals and define the densities of states, $\sigma(k)$ and $\sigma_0(k_0)$, by the relations $D(\delta k) = \sigma(k)\delta k$ and $D(\delta k^{(0)}) = \sigma_0(k^{(0)})\delta k^{(0)}$, where $D(\delta k)$ and $D(\delta k^{(0)})$ are the numbers of states in the small intervals δk and $\delta k^{(0)}$, respectively. If δk and $\delta k^{(0)}$ are related by Eq. (19), then $D(\delta k) = D(\delta k^{(0)})$. Obviously, $\sigma_0(k_0) = L/(2\pi)$ (if we impose periodic boundary conditions on $k^{(0)}$) [22, 23]. In accordance with the notations in the literature [20, 23] I introduce here also the particle density,

$$\rho(k) = \sigma(k)n(k) / L = N(\delta k) / (L\delta k).$$

In the new notations, Eq. (19) becomes a self-consistent equation for k ,

$$k = k^{(0)}(k) + \int \theta(k - k') \rho(k') dk'$$

from which we can calculate [23] $dk^{(0)}/k = 1 - \int \theta'(k-k')\rho(k')dk'$, where $\theta'(k) = d\theta(k)/dk$. From here we get the density of states $\sigma(k) = [L/(2\pi)] \{1 - \int \theta'(k-k')\rho(k')dk'\}$

The FES is manifesting in the system because of the dependence of σ on ρ : the variation of $\rho(k_i)$ produces, in principle, a change of the density of states $\sigma(k)$, at any k . Applying the FES formula, $\delta D(\delta k) = -\alpha_{\delta k \delta k_i} \delta N(\delta k_i)$, in Ref. [7] I obtained

$$\alpha_{\delta k \delta k_i} = \frac{1}{2\pi} \theta'(k - k_i) \delta k \quad (20)$$

and therefore the exclusion statistics parameter $\alpha_{\delta k \delta k_i}$ is proportional to the dimension of the space on which it acts, $D(\delta k)$.

5. CONCLUSIONS

This paper is a short review of the main results that I obtained recently related to the fractional exclusion statistics (FES). After a short introduction to FES, made in Section 2, I pointed out that FES needs to be amended in order to make it a self-consistent theory. These amendments were presented in Section 3 and are expressed mathematically by Eqs. (10).

In Section 4 I introduced an ansatz for the FES parameters and I applied FES to two well known models of systems of particles with interactions. By calculating the FES parameters in these examples, I showed that the ansatz proposed in the beginning is satisfied.

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