ORIGINAL PAPER

BANACH SPACE VALUED POTENTIAL TRANSFORM

NANDA D. SAHU[1], ALKA S. GUDADHE[2]

Manuscript received: 07.05.2011. Accepted paper: 30.05.2011.

Published online: 10.06.2011.

Abstract. *This paper studies different testing function spaces of Gelfand-Shilov type for the Banach space valued generalized Potential transform. The topological properties of these spaces are discussed. Different operators and their continuity is also discussed. The Analyticity theorem and Inversion theorem for Banach space valued distributional Potential Transform are also proved.*

Keywords: *generalized function, Banach space, Potential transform, testing function spaces.*

1. INTRODUCTION

Zemanian [4] has presented the theory of Banach space valued distribution. He has also discussed the Laplace transform of Banach space valued distribution. Further he has used these concepts for applications in the system theory and signals.Motivated by the work of Zemanian [4] and Tiwari [1, 2] we studied different testing function spaces of Gelfand-Shilov type for the the Banach space valued Potential transform. The topological properties of these spaces, different operators and their continuity and the Analyticity theorem are discussed. We also state and prove the inversion formula for Banach space valued Potential transform.

2. BANACH SPACE ALUED TESTING FUNCTION SPACES

2.1. THE SPACES $P_{c,d,\alpha}(A)$:

Let A be a Banach space. For $\alpha \geq 0$, $P_{c,d,\alpha}(A)$ is defined as,

$$P_{c,d,\alpha}(A) = \left\{ \psi : \psi \in E_+(A); i_{c,d,k}(\psi) = \sup_{0<t<\infty} \left\| \lambda_{c,d}(t)(tD_t)^k \{t\psi(t)\} \right\|_A \leq C_k L^k k^{k\alpha}, k = 0,1,2,3... \right\}$$

The constant L and C_k depend on the function ψ and

$$\lambda_{c,d}(t) = \begin{cases} t^c, & 1 \leq t < \infty, \\ t^d, & 0 < t < 1, \end{cases} \quad (1)$$

---

[1] Sipna's College of Engineering and Technology, 444701, Amravati, India.
E-mail: nandasahu31@gmail.com.
[2] Government Vidarbha Institute of Science and Humanties, Department of Mathematics, 444701, Amravati, India. E-mail: alka.gudadhe@gmail.com.

where *c* and *d* are real numbers. For $k = 0$, we set $k^{k\alpha} = 1$.

The topology of the space $P_{c,d,\alpha}(A)$ is generated by the family of seminorms $\{i_{c,d,k}\}_{k=0}^{\infty}$ denoted by $T_{c,d,\alpha}(A)$.

2.2. THE SPACES $P_{c,d,\alpha,m}(A)$:

For given $m > 0$, the space is defined as,

$$P_{c,d,\alpha,m}(A) = \left\{\psi : \psi \in E_{+}(A); i_{c,d,k}(\psi) = Sup\left\|\lambda_{c,d}(t)(tD_t)^k\{t\psi(t)\}\right\|_A \le C_{k\delta}(m+\delta)^k k^{k\alpha}, k = 0,1,2,3...\right\}.$$

The constant $\delta > 0$, $C_{k\delta}$ depend on k, δ and the function ψ, $\lambda_{c,d}(t)$ is as in equation (1).

The topology of the $P_{c,d,\alpha,M}(A)$ is generated by the family of seminorms $\{i_{c,d,k}\}_{k=0}^{\infty}$ and denoted by $T_{c,d,\alpha,m}(A)$. Clearly the space $P_{c,d,\alpha,m}(A)$ is subspace of $P_{c,d,\alpha}(A)$.

2.3. THE SPACES $P(w,z;A)$:

Following Zemanian [5, p. 102], the space $P(w,z;A)$ is defined as,

$P(w,z;A) = \underset{n\to\infty}{ind}\, P_{c_n,d_n}(A)$ where $c_n \to w_+,\ d_n \to z_-,\ w,z \in [-\infty,\infty]$.

3. TOPOLOGICAL PROPERTIES OF TESTING FUNCTION SPACES

3.1. PROPOSITION:

If $m_1 < m_2$ then $p_{c,d,\alpha,m_1}(A) \subset p_{c,d,\alpha,m_2}(A)$ and $T_{a,b,\alpha,m_1}(A) \approx T_{a,b,\alpha,m_2}(A) / T_{a,b,\alpha,m_1}(A)$

Proof is simple and hence omitted.

3.2. PROPOSITION:

$$P_{c,d,\alpha}(A) = \bigcup_{i=1}^{\infty} P_{c,d,\alpha,m_i}(A)$$

Proof: We know that $P_{c,d,\alpha,m_i}(A) \subseteq P_{c,d,\alpha}(A)$, for each $m_i, i = 1,2,...$

$$\Rightarrow \bigcup_{i=1}^{\infty} P_{c,d,\alpha,m_i}(A) \subseteq P_{c,d,\alpha}(A) \qquad (2)$$

Now let us consider $\psi \in P_{c,d,\alpha}(A)$.

Therefore, we have

$$i_{c,d,k}(\psi) = \underset{0<T<\infty}{Sup}\left\|\lambda_{c,d}(t)(tD_t)^k\{t\psi(t)\}\right\|_A \le C_k L^k k^{k\alpha},\ k = 0,1,2,... \qquad (3)$$

where L, C_k are depends on the function ψ

Choose an integer m_i for some $i=1,2,3\ldots$ and $\delta > 0$ such that

$$C_k L^k k^{k\alpha} \le C_{k\delta}(m_i + \delta)^k k^{k\alpha}.$$

Using equation 3 it immediately follows that,

$$i_{c,d,k}(\psi) \le C_{k\delta}(m_i + \delta)^k k^{k\alpha} \text{ for some } m_i$$

$$\Rightarrow \psi \in P_{c,d,\alpha,m_i}(A) \Rightarrow \psi \in \bigcup_{i=1}^{\infty} P_{c,d,\alpha,m_i}(A) \Rightarrow P_{c,d,\alpha}(A) \subseteq P_{c,d,\alpha,m_i}(A) \qquad (4)$$

From equations (2) and (4) we have, $P_{c,d,\alpha}(A) = \bigcup_{i=1}^{\infty} P_{c,d,\alpha,m_i}(A)$.

3.3. PROPOSITION:

If $\alpha_1 < \alpha_2$ then $P_{c,d,\alpha_1}(A) \le P_{c,d,\alpha_2}(A)$.

Proof is simple and hence omitted.

3.4. THEOREM:

The space *D(A)* is subspace of $P_{c,d,\alpha}(A)$ and cannonical injection of *D(A)* into $P_{c,d,\alpha}(A)$ is continuous.

Proof: Let $\psi(t) \in D(A) \Rightarrow \psi(t)$ is a smooth function whose support is bounded set K of $(0,\infty)$.

Let $L_l = \underset{K}{Sup}\{(t) : t \in Sup\,\psi\}$, $b_k = \underset{K}{Sup}\{\lambda_{c,d}(t)(D_t)^k\{t \in \psi(t)\}\}$.

Then $i_{c,d,k}(\psi) = \underset{K}{Sup}\left\|\lambda_{c,d}(t)(tD_t)^k\{t\psi(t)\}\right\|_A \le b_k L_l^{\ k} \le b_k L_k\ .\ k^{k\alpha}$.

Hence $\psi \in P_{c,d,\alpha}(A)$. Therefore $D(A) \subset P_{c,d,\alpha}(A)$.

To prove contiuity if the injection mapping, consider a sequence converging to zero in *D(A)*.

As $\psi_n \in D(A)$,we can find a compact set K_m such that $\psi_n \in D_{K_m}(A)$ and $\psi_n \to 0$ in $D_{K_m}(A)$

$$i_{c,d,k}(\psi) = \underset{t}{Sup}\left\|\lambda_{c,d}(t)(tD_t)^k\{t\psi(t)\}\right\|_A$$

Let $C_k = \underset{t}{Sup}\left\|\lambda_{c,d}(t)\right\|_A$ then $i_{c,d,k}(\psi) \le C_k\ Sup\left\|(tD_t)^k\{t\psi(t)\}\right\|_A$.

The right hand side of the above inequality tends to zero as $\psi_n \to 0$ in $D(A)$ and hence cannonical injection of $D(A)$ into $P_{c,d,\alpha}(A)$ is continuous.

4. OPERATORS ON THE TESTING FUNCTION SPACE $P_{c,d,\alpha}(A)$

4.1. SHIFTING OPERATORS ON THE TESTING FUNCTION SPACE $P_{c,d,\alpha}(A)$

Theorem: Shifting operator, $S:\psi(t)\rightarrow\psi(t+\xi)$ is automophism on $P_{c,d,\alpha}(A)$.

Proof: Here $S:\psi(t)\rightarrow\psi(t+\xi)$ is well defined and linear.

For continuity consider,

$$i_{c,d,k}[\phi(t+\xi)] = \sup_{t}\left\|\lambda_{c,d}(t)(tD_t)^k\{t\phi(t+\xi)\}\right\|_A$$

$$= \sup_{1\le t<\infty}\left\|\lambda_{c,d}(T-\xi)(T-\xi)^k(D_t)^k\{(T-\xi)\phi(T)\}\right\|_A$$

$$\le C_k^{'}L^k\left(\sum_{i=1}^{k}C_iD_T(T-\xi)D_T^{k-1}\phi(T)\right) \le C_k^{'}L^k\left(\sum_{i=1}^{k}C_iD_T^{k-1}\phi(T)\right)$$

here *C'* is some constant. Theefore the shifting operator $S:\psi(t)\rightarrow\psi(t+\xi)$ is a topological automorphism on the space $P_{c,d,\alpha}(A)$.

4.2. SCALING OPERATOR ON THE TESTING FUNCTION SPACE $P_{c,d,\alpha}(A)$

Theorem: Scaling operator, $S:\psi(t)\rightarrow\psi(mt)$ is automophism on $P_{c,d,\alpha}(A)$.

Proof: Here $S:\psi(t)\rightarrow\psi(mt)$ is well defined and linear.

For continuity consider,

$$\sup_{t}\left\|\lambda_{c,d}(t)(tD_t)^k\{t\psi(t)\}\right\|_A = \sup_{1\le t<\infty}\left\|\left(\frac{T}{m}\right)^{C+k}D_t^k\left\{\frac{T}{m}\phi(T)\right\}\right\|_A \le C_kL^kk^{k\alpha} \qquad (5)$$

Where C_k is some constant. We have $\psi(mt)\in P_{c,d,\alpha}(A)$.

Therefore scaling operator, $S:\psi(t)\rightarrow\psi(mt)$ is automophism on $P_{c,d,\alpha}(A)$.

4.3. INVERSE SCALING OPERATOR ON THE TESTING FUNCTION SPACE $P_{c,d,\alpha}(A)$

Theorem: Scaling operator, $S^{-1}:\psi(t)\rightarrow\psi\left(\frac{t}{m}\right)$ is automophism on $P_{c,d,\alpha}(A)$.

Proof: The proof is simple as 4.2. and hence omitted.

4.4. DIFFERENTIAL OPERATOR ON THE TESTING FUNCTION SPACE $P_{c,d,\alpha}(A)$

Theorem: The differential operator, $\Delta : P_{c+\gamma,d+\gamma,\alpha}(A) \to P_{c,d,\alpha}$ difined by the map $\Delta\psi = D_t^r\psi = \phi$ is one one, linear and continuous.

Proof: Let $\psi \in P_{c+\gamma,d+\gamma,\alpha}(A)$ then

$$\begin{aligned} i_{c,d,k}\left(D_t^r\psi\right) &= \underset{t}{Sup}\left\|\lambda_{c,d}(t)(tD_t)^k\left\{tD_t^r\psi(t)\right\}\right\|_A = \underset{t}{Sup}\left\|\lambda_{c+r,d+r}(t)t^k t^r D_t^k\left\{t\phi(t)\right\}\right\| \\ &= \underset{t}{Sup}\left\|\lambda_{c+r,d+r}(t)t^{k+r}\left(\sum_{i=1}^{k} C_i D_t t D_t^{k-1}\phi(t)\right)\right\| \le C_k L^{k+r}\left(\sum_{i=1}^{k} C_i D_t^{k+r-1}\phi(t)\right) \end{aligned} \tag{6}$$

Thus $D^r\phi \in P_{c,d,\alpha}(A)$ and therefore operator Δ is well defined.

Linearity of the operator is obvious.

It is injective for $\Delta\phi = 0 \Rightarrow D^r\phi = 0$ for $r = 1$ $\phi = c$, where c is a constant.

If $c = 0$, $\phi = 0$ otherwise $\phi = c$. For $k = 0,\ r = 0$ we have

$$\underset{t}{Sup}\left\|\lambda_{c+r,d+r}(t)t^k t^r D_t^k\left\{t\phi(t)\right\}\right\|_A = \underset{t}{Sup}\left\|\lambda_{c,dr}(t)C\right\|_A$$

As the right hand side of the above equation is not bounded, we conclude that $\phi \notin P_{c+r,d+r,\alpha}(A)$. Wich is a contradiction.

Hence C must be zero and therefore $\phi = 0$. This proves that Δ is injective continuity of Δ. Follows from the inequality equation (6). Hence proved.

5. BANACH SPACE VALUED DISTRIBUTIONAL POTENTIAL TRANSFORM

5.1. POTENTIAL TRANSFORM OF A-VALUED DISTRIBUTION

Let $f \in [D_+; A]$ i.e. f is a function defined on D_+ to A.

f is said to be Banach space valued Potential transformable, if there exists two members $\sigma_1, \sigma_2 \in [-\infty; \infty]$, such that $\sigma_1 < \sigma_2$, $f \in [P(\sigma_1, \sigma_2); A]$ and in addition $f \notin [P(w, z); A]$ if either $w < \sigma_1$ or $z < \sigma_2$.

With $\Omega_f = \{y : \sigma_1 < \mathrm{Re}(y) < \sigma_2\}, \dfrac{\mathrm{t}}{\mathrm{y}^2 + \mathrm{t}^2} \in P(\sigma_1, \sigma_2)$. Then the Potential transform is defined as $F(y) = \left\langle f(t), \dfrac{t}{t^2 + y^2} \right\rangle, y \in \Omega_f$. $F(y)$ is an A-valued analytic function on Ω_f.

5.2. POTENTIAL TRANSFORM OF THE [A;B] VALUED DISTRIBUTION

The space $[P(w,z;A);B]$ can be identified with the space $\{[P(w,z);\ [A;\ B]\}$ through the equation, $\langle f_p,\phi\rangle a=\langle p,\phi a\rangle$ where $f_p\in\{P(w,z);\ [A;\ B]\}$, $p\in[P(w,z;\ A);\ B]$, $\phi\in P(w,z)$ and $a\in A$.

Because of the above identification we use the same symbol to denote both f_p and p and define Potential transform of $[A;B]$ valued distribution as:

$p\in[D(A);B]$ is said to be Potential transformable if there exist $\eta_1\eta_2\in[-\infty,\infty]$, such that $\eta_1<\eta_2$, $p\in[P(\eta_1,\eta_2;A);B]$ and $p\notin[P(w,z;A);B]$, if either $w<\eta_1$ or $z<\eta_2$.

6. ANALYTICITY THEOREM

Theorem: If $P[f(t)]=F(y)$ for $y\in\Omega_f$, where $\Omega_f=\{y:\sigma_1<\operatorname{Re}(y)<\sigma_2\}$ and $F\in[D(A),B]$ then $F(y)$ is an $[A;B]$ valued analytic function on Ω_f, where $y\in\Omega_f$ for each nonnegative integer k, $DF(y)=\left\langle f(t),\frac{t}{t^2+y^2}\right\rangle$, i.e. $DF(y)=\langle f(t),K(y,t)\rangle$.

Also $D^qF(y)=\langle f(t),K_q(y,t)\rangle$, where $K_q(y,t)=\frac{\partial^q}{\partial t^q}K(y,t)$ and $K(y,t)=\frac{t}{t^2+y^2}$.

Proof: $DF(y)=\langle f(t),K(y,t)\rangle$ has meaning, since $f\in[P(\sigma_1,\sigma_2);A]$ and $\frac{t}{t^2+y^2}\in P(\sigma_1,\sigma_2)$.

Let y be an arbitrary but fixed point in Ω_f. Choose the real positive numbers, a,b,r and r_1 such that $\sigma_1<a<\operatorname{Re}(y-r)<\operatorname{Re}(y-r_1)<b<\sigma_2$ also Δy be the complex increament such that $|\Delta y|<r$. Consider,

$$\frac{F(y+\Delta y)-F(y)}{\Delta y}-\left\langle f(t),\frac{\partial}{\partial y}\frac{2}{\pi}\frac{t}{y^2+t^2}\right\rangle=\frac{2}{\pi}\langle f(t),\psi_{\Delta y}(t)\rangle,$$

where

$$\psi_{\Delta y}(t)=\frac{1}{\Delta y}\left[\frac{t}{(y+\Delta y)^2+t^2}-\frac{t}{y^2+t^2}\right]-\frac{\partial}{\partial y}\frac{t}{y^2+t^2},$$

which can be written as,

$$\psi_{\Delta y}(t)=\frac{1}{\Delta y}[K(y+\Delta y,t)-K(y,t)]-\frac{\partial}{\partial y}K(y,t),$$

The theorem is proved for *q = 1* and generalized it by Induction method,

$$\therefore D^q\{\psi_{\Delta y}(t)\}=\frac{1}{\Delta y}[K_q(y+\Delta y,t)-K_q(y,t)]-\frac{\partial}{\partial y}K_q(y,t),$$

where

$$K_q(y,t)=\frac{\partial^q}{\partial t^q}K(y,t).$$

To proceed, let c denotes the circle with the centre y and radius r_1.

Let us restrict r_1 such that c lies entirely with Ω_f and $0 < r < r_1$.

The differentiation on y is interchanged with differentiation on t and using Cauchy's integral formula we get,

$$\therefore D^q\{\psi_{\Delta y}(t)\} = \frac{1}{2\pi i\Delta y}\int_c \left[\frac{1}{\xi-(y+\Delta y)} - \frac{1}{\xi - y}\right] K_q(\xi,t)d\xi - \frac{1}{2\pi i}\int_c \frac{k_q(\xi,t)}{(\xi-y)^2}d\xi\ ,\ \xi \in c$$

therefore, $D_t^q\{\psi_{\Delta y}(t)\} = \frac{\Delta y}{2\pi i}\int_c \frac{k_q(\xi,t)}{(\xi-y-\Delta y)(\xi-y)^2}d\xi$.

Now for fixed $\zeta \in c, -\infty < t < \infty, \left|\lambda_{a,b}(t)K_q(\xi,t) \le N\right|$ for some constant N.

Moreover, $|\xi - y - \Delta y| > r_1 - r > 0$ and $|\xi - y| = r_1$

$$\left|\rho_{a,b,c}\psi_{\Delta y}(t)\right| \le \frac{|\Delta y|}{2\pi}\int_c \frac{|N|}{\left|\xi - y^2\right|\left|\xi - y - \Delta y\right|}d\xi \le \frac{|\Delta y|}{r^2(r_1 - r)}|N| \to 0 \text{ as } |\Delta y| \to 0.$$

This shows that $\psi_{\Delta y}(t) \to 0$ as $|\Delta y| \to 0$ and hence the proof is complete.

7. INVERSE OF BANACH SPACE VALUED POTENTIAL TRANSFORM

Sahu in [3] has presented that Banach Space Value Potential transform is iterated second version Laplace transform, provided it exist, i.e.

$$[L_2\{L_2 f(t)\}(s)][y] = \frac{1}{2}[Pf(t)](y)$$

Using the definition of second version Laplace transform we obtain,

$$[L_2\{L_2 f(t)\}(s)][y] = \left\langle\left\langle f(t); te^{-t^2s^2}\right\rangle; ze^{-y^2s^2}\right\rangle.$$

7.1. LEMMA:

Let $F(\sqrt{s})$ is analytic function of s (assuming that $s = 0$ is not a branch point) except at finite number of poles each of which lies to the left of the vertical lines $\operatorname{Re} s = c$ and if $F(\sqrt{s}) \to 0$ as $s \to \infty$ throught the left plane $\operatorname{Re} s \le c$, suppose that:

$$[L_2 f(t)](s) = \left\langle f(t); te^{-t^2s^2}\right\rangle = F(s),\ L_2^{-1}\{F(s)\} = \frac{1}{2\pi i}\left\langle 2F(\sqrt{s}); e^{st^2}\right\rangle = f(t)$$

Proof: By Definition, $[L_2 f(t)](s) = \left\langle f(t); te^{-t^2s^2}\right\rangle = F(s)$.

Put $s^2 = r$ $F(\sqrt{r}) = \left\langle f(t); te^{-t^2r}\right\rangle$. Let $t^2 = x$ $F(\sqrt{r}) = \frac{1}{2}\left\langle f(\sqrt{x}); te^{-xr}\right\rangle$.

Also, complex inversion formula for Laplace Transform is

$$f(\sqrt{x}) = \frac{1}{2\pi i}\left\langle 2F(\sqrt{r}); e^{xr}\right\rangle.$$

If we set $t^2 = x$, we obtain $f(t) = \frac{1}{2\pi i}\left\langle 2F(\sqrt{r}); e^{rt^2}\right\rangle$

If we replace *r* by *s*, we get, $f(t)=\frac{1}{2\pi i}\left\langle 2F(\sqrt{s}), e^{st^2}\right\rangle$.

7.2. COMPLEX INVERSION THEOREM:

Let $f\in[D(A);B]$ and $P\{f(t),y\}=F(y)$, for $y\in\Omega_f=\{y:\sigma_1<\operatorname{Re}(y)<\sigma_2\}, \frac{t}{y^2+t^2}\in P(\sigma_1,\sigma_2)$ then in the sense of convergence in $[D(A);B]$

$$f(t)=\frac{1}{4\pi i}\left\langle \frac{2}{2\pi i}\left\langle 2F(\sqrt{y}), e^{yw^2}\right\rangle_{w\to w^2} e^{wt^2}\right\rangle$$

provide that, all integrals involved are convergent.

Proof: Let $\phi\in D(A)$. Choose $\sigma_1\prec a\prec b\prec\sigma_2$.

We want to show that $\langle f(t),\phi(t)\rangle=\frac{1}{4\pi i}\left\langle \frac{2}{2\pi i}\left\langle 2F(\sqrt{y}), e^{yw^2}\right\rangle_{w\to w^2} e^{wt^2}\right\rangle$.

That is the natural consequence of definition of Potential transform and Complex Inversion Formula for Second version Laplace transform.

REFERENCES

[1] Tiwari, A. K., *Indian Journal of Pure and Applied Mathematics*, **11**(8), 1045, 1980.
[2] Tiwari, A. K., *Indian Journal of Pure and Applied Mathematics*, **20**(5), 493, 1989.
[3] Sahu N. D., Gudadhe, A. S., *Acta Ciencia Indica*, **35**(2), 803, 2009.
[4] Zemanian, A. H., *Realizability Theory for Continuous Linear System*, Academic Press, New York, 1972.