ORIGINAL PAPER

A NOTE ON REAL LIPSCHITZ FUNCTIONS OF ONE VARIABLE

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Abstract. *In this paper we give two proofs for a result regarding the antiderivatives of functions of the form* $g(x) = f(x) - \alpha x$, *where* $f:[0,1] \to \mathbb{R}$ *is a Lipschitz function.*

Keywords: *antiderivative, Lipschitz function.*

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1. INTRODUCTION

In [1] it proves that every Lipschitz function $F:\mathbb{R} \to \mathbb{R}$, $|F(x) - F(y)| \le m|x-y|$ for all $x, y \in \mathbb{R}$, can be written as sum of two injective Lipschitz functions, $F = F_1 + F_2$ ($F_1:\mathbb{R} \to \mathbb{R}$, $F_2:\mathbb{R} \to \mathbb{R}$). In fact, $F_1 = F - aI$, where I is the identity of $\mathbb{R}$ and $a > m$.

Let $f:[0,1] \to \mathbb{R}$ be a Lipschitz function, $|f(x) - f(y)| \le k|x-y|$ for all $x, y \in \mathbb{R}$ ($k > 0$) and $g:[0,1] \to \mathbb{R}$ defined by $g(x) = f(x) - (k+\theta)x$, $\theta > 0$. It is clear that g has antiderivatives, being continuous on $[0,1]$.

In this note we prove a result about the antiderivatives of the function g. We give two proofs for this result, the second using the same type of reasoning as in the proof of the said result from paper [1].

2. RESULTS

Theorem 2.1 If $f(0) = 0$, then $G(0) \ne G(1)$ for all G antiderivative of g.

First Proof: We have $|f(x)| \le k|x|$ for all $x \in [0,1]$. Consequently we obtain

$$\left|\int_0^1 f(x)dx\right| \le \int_0^1 |f(x)|dx \le \int_0^1 kxdx = \frac{k}{2}.$$

Suppose that it exists G, an antiderivative of g, so that $G(0) = G(1)$. Then

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$$\int_0^1 g(x)dx = 0 .$$

It results that

$$\int_0^1 f(x)dx = \int_0^1 (k+\theta)xdx .$$

Therefore

$$\int_0^1 f(x)dx = \frac{k+\theta}{2}$$

and this fact is contradictory to $\left|\int_0^1 f(x)dx\right| \le \frac{k}{2}$.

Second Proof: Let $x \in [0,1]$. We have

$$(k+\theta)|x-y| = |(f(x)-g(x))-(f(y)-g(y))| \le$$
$$|f(x)-f(y)|+|g(x)-g(y)| \le k|x-y|+|g(x)-g(y)| .$$

It results now that $|g(x)-g(y)| \ge \theta|x-y|$ for all $x \in [0,1]$ and consequently g is injective. $g(0) = f(0) = 0$ implies that $g(x) \ne 0$ for all $x \in (0,1]$. If it exists G, an antiderivative of g, so that $G(0) = G(1)$, then we have a $\xi \in (0,1)$ with $g(\xi) = 0$ and this is contradictory to the fact that g is not null on $(0,1)$. Thus the second proof of Theorem 2.1 is complete.

REFERENCE

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