ORIGINAL PAPER

PROPAGATORS AND DILATIONS ON PSEUDO-HILBERT SPACES

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Manuscript received: 21.02.2011. Accepted paper: 04.04.2011.

Published online: 10.06.2011.

Abstract. *In this paper we shall try to transpose the conditions of the existence of propagators for kernel on *-semigroups, notion introduced by P. Masani [6]. We will start with a few additional observations concerning *-representations and then we will present some properties of propagators and dilations.*

Keywords: *pseudo-Hilbert spaces, *-representations, propagators, dilations.*

2000 Mathematics subject classification: *47 A 45; 42 B 10.*

1. INTRODUCTION

In order to prove the results of the following sections, we need to recall the next definitions and properties. Let **Z** - be an admissible space and $\mathcal{H}$ a Loynes **Z-** space, see [1, 4, 5].

Lemma 1. [1] If p is a continuous and monotone seminorm on **Z**, then

$$q_p(h) = (p([h,h]))^{\frac{1}{2}}$$

is a continuous seminorm on $\mathcal{H}$.

Proposition 1. [1] If $\mathcal{H}$ is a pre-Loynes **Z**-space and $\mathcal{P}_Z$ is a set of monotonous (increasing) seminorms defining the topology of **Z**, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathbf{P}} = \{q_p \mid p \in \mathcal{P}_Z\}$.

Consequence 1. [1] Using the above notations, for every monotone seminorm p on **Z** the following inequality holds:

$$p([h,k]) \leq 2q_p(h)q_p(k) \text{ for all } h, k \in \mathcal{H}.$$

We say that an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is in $\mathcal{C}(\mathcal{H}, \mathcal{K})$ if and only if for every seminorm q_p^2 on $\mathcal{K}$, there exists a constant $M_p > 0$ and a seminorm $q_{p_0}^1$ on $\mathcal{H}$ such that

$$q_p^2(Th) \leq M_p q_{p_0}^1(h), h \in \mathcal{H}.$$

Obviously, this condition will be equivalent with the condition: for every seminorm $p \in \mathcal{P}_Z$, there is a constant $M_p > 0$ and a seminorm $p_0 \in \mathcal{P}_Z$ such that

$$p([Th,Th]_{\mathbf{K}}) \leq M_p^2 p_0([h,h]_{\mathbf{H}}), \quad h \in \mathcal{H}. \tag{1}$$

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If we take above $p_0 = p$ then we obtain the class $\mathcal{CQ}(\mathcal{H}, \mathcal{K})$ and $\mathcal{CQ}^*(\mathcal{H}, \mathcal{K}) = \mathcal{CQ}(\mathcal{H}, \mathcal{K}) \cap \mathcal{L}^*(\mathcal{H}, \mathcal{K})$. Also, we recall that an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called gramian bounded ($T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$), if there exists a constant $\mu > 0$ such that in the sense of order of Z, the following inequality holds

$$[Th, Th]_{\mathbf{K}} \leq \mu\, [h, h]_{\mathbf{H}}, \ h \in \mathcal{H}. \tag{2}$$

The study of Loynes Z-spaces with a given reproducing kernel was given in [3] Theorem 4.1. We will state a similar result for an arbitrary Z-space. The notions of reproducing kernel and sesquilinear Z-form are also given in [2].

Theorem 1. Let Z be an admissible space in the Loynes sense. For any positive definite kernel Γ. There is a unique Loynes Z-space $\mathcal{H}_\Gamma$, which admits Γ as a reproducing kernel.

Definition 1. Let Z be an admissible semigroup and $\Gamma: S\times S \to Z$ a Z-valued kernel on S. Γ satisfies the boundedness condition, if there is a function $c: S \to [0,\infty)$ so that

$$c(u)\Gamma - \Gamma_u \tag{BC}$$

is positive definite for all $u \in S$, where $\Gamma_u(s,t) = \Gamma(us, ut)$.

Γ satisfies the q-boundedness condition (BCQ), if for every seminorm $p \in \mathcal{P}_Z$, there is a function $c_p: S \to [0,\infty)$ such that

$$p\Big(\sum_{j,k=1}^{n} c_j \overline{c_k} \Gamma_u(s_j, s_k)\Big) \leq c_p(u)\, p\Big(\sum_{j,k=1}^{n} c_j \overline{c_k} \Gamma(s_j, s_k)\Big) \tag{BCQ}$$

for all $n \in \mathbf{N}$, $c_1,\dots,c_n \in \mathbf{C}$, $s_1,\dots,s_n \in S$, $u \in S$.

Γ will satisfy the continuity condition (CC), if for every seminorm $p \in \mathcal{P}_Z$, there are two functions on S, $\gamma_p: S \to \mathcal{P}_Z$ and $c_p: S \to [0,\infty)$ such that

$$p\Big(\sum_{j,k=1}^{n} c_j \overline{c_k} \Gamma_u(s_j, s_k)\Big) \leq c_p(u)\gamma_p(u)\Big(\sum_{j,k=1}^{n} c_j \overline{c_k} \Gamma(s_j, s_k)\Big) \tag{CC}$$

for all $n \in \mathbf{N}$, $c_1,\dots,c_n \in \mathbf{C}$, $s_1,\dots,s_n \in S$, $u \in S$.

Definition 2. If C is a $\mathcal{F}(\mathcal{H}, Z)$ –valued kernel on the semigroup S, then C satisfies:

(i) the boundedness condition, if there exists a function $\rho: S \to [0,\infty)$ such that

$$\rho(u)C - C_u \text{ is positive definite } (u \in S), \tag{BC}$$

where $C_u(s,t) = C(us, ut)$.

(ii) q-boundedness condition (BCQ), if the associate Z-valued kernel $\Gamma=\Gamma_C$ defined by $\Gamma_C(\lambda,\mu)=C(t,s)(k,h), \lambda=(s,h), \mu=(t,k)\in S\times$ $\mathcal{H}$ satisfies for a function c_p(u), the condition (BCQ) from Definition 1, where we consider $\Gamma_u=\Gamma_{C_u}$;
(iii) the continuity condition (CC), if there exist the functions c_p:S$\to[0,\infty)$ and γ_p:S$\to$ $\mathcal{P}_Z$ such that the conditions (CC) from the Definition 1 with Γ_C and Γ_{C_u} instead of Γ and Γ_u respectively take place.

Definition 3. The function $\mathcal{F}(\mathcal{H}$, Z) –valued ϕ, defined on the $*-$ semigroup S is called positive definite if the $\mathcal{F}(\mathcal{H}$, Z) –valued associated kernel $C_\phi : S\times S\to$ $\mathcal{F}(\mathcal{H}$, Z) defined by $C_\phi(s,t)=\phi(t$ *s), s, t$\in$ S is positive definite.

We say that such a function ϕ satisfies the boundedness conditions (BC), (BCQ), (CC) respectively, if the associated kernel C_ϕ satisfies the corresponding conditions from the Definition 2.

2. PSEUDO-HILBERT REPRESENTATIONS OF $*-$ SEMIGROUPS

Given a Loynes Z-space $\mathcal{H}$, we can associate with it the algebra of the linear operators $\mathcal{L}(\mathcal{H})$ and the involutive sub-algebra $\mathcal{L}*(\mathcal{H})$, respectively. Looking now at the different types of continuities in $\mathcal{L}(\mathcal{H})$, we can identify in a decreasing order the sub-algebras $\mathcal{C}(\mathcal{H})$, $\mathcal{CQ}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$.

Let us recall now that [$\mathcal{B}$ * ($\mathcal{H}$)]*$\subset$ $\mathcal{B}$ * ($\mathcal{H}$). But it isn't sure that [$\mathcal{C}$ * ($\mathcal{H}$)]*$\subset$ $\mathcal{C}$ * ($\mathcal{H}$). Concerning the sub-algebras, we can state the following:
Remark 1 The adjoint of any linear operator (if there is one) q--bounded remains q-bounded. More precisely, the following inclusion takes place for any Loynes Z-space.

$$[\mathcal{C}\mathcal{Q}^* (\mathcal{H})]^*\subset \mathcal{CQ}^* (\mathcal{H}). \tag{3}$$

Indeed, if T$\in\mathcal{C}\mathcal{Q}$* ($\mathcal{H}$) and $p\in \mathcal{P}_Z$, we shall denote by M_p the positive constant, for which $q_p(Th)\le M_p q_p(h), h\in$ $\mathcal{H}$, and then applying the definition of q_p and the Schwarz type inequality from Consequence 1, the following inequalities occur successively for $q_p(T^*h)\neq 0$:

$$[q_p(T^*h)]^2 = p([T^*h,T^*h]) = p([TT^*h,h])\le$$
$$\le 2p([TT^*h,TT^*h])^{1/2}p([h,h])^{1/2} = 2q_p(TT^*h)q_p(h)\le 2M_p q_p(T^*h)q_p(h),$$

hence $q_p(T^*h)\le 2M_p q_p(h),$ inequality which obviously will be also checked for h for which $q_p(T^*h)=0.$

According to Definition 2, (ii) the Z-valued associated kernel $\Gamma=\Gamma_C$ can be defined by $C(s,t)(h,k)=\Gamma_C(\lambda,\mu);\lambda=(t,k),\mu=(s,h)\in S\times$ $\mathcal{H}$. In this case the positivity condition of C becomes:

$$\sum_{j,l=1}^{n} C(s_l,s_j)(h_l,h_j)=\sum_{j,l=1}^{n}\Gamma_C(\lambda_j,\lambda_l)\ge 0$$

where $\lambda_j = (s_j, h_j), \lambda_l = (s_l, h_l)$. The kernel Γ_C is positive definite iff C is a $\mathcal{F}(\mathcal{H}$, Z)-valued kernel on S$\times \mathcal{H}$, and Γ_C will be linear in the second variable and anti-linear in the first variable.

Definition 4. Any algebraic morphism of a semigroup with values in one of the previously defined operator algebras on a certain Loynes Z-space is called a pseudo-Hilbert representation of the given semigroup. More precisely, if S is a semigroup, $\mathcal{H}$ a Loynes Z-space and $\mathcal{G}(\mathcal{H})$ is one of the operator algebras (i.e. the position of $\mathcal{G}$ is successively taken by $\mathcal{L}$, $\mathcal{C}$, $\mathcal{CQ}$, $\mathcal{B}$), then $\pi : S \to \mathcal{G}(\mathcal{H})$ is a representation on $\mathcal{H}$ if

$$\pi(st) = \pi(s)\pi(t); \quad s, t \in S. \tag{4}$$

π is called unital pseudo-Hilbert representation, $\mathcal{G}(\mathcal{H})$-valued, if the semigroup S has unit e and π satisfies even more

$$\pi(e) = I_{\mathbf{H}}. \tag{5}$$

π is called *-pseudo-Hilbert representation if the values of the morphism are in one of the algebras $\mathcal{G}^*(\mathcal{H})$ with $\mathcal{L}$, $\mathcal{C}$, $\mathcal{CQ}$, $\mathcal{B}$ in the position of $\mathcal{G}$, S is a *-semigroup and π satisfies

$$\pi(s^*) = \pi(s)^*, \quad s \in S. \tag{6}$$

Now, we shall refer to the positivity and boundedness properties which are satisfied by the pseudo-Hilbert representations. These properties will be formulated for a representation π using the language of the $\mathcal{F}(\mathcal{H}$, Z) –valued kernel C_π associated to π by

$$[C_\pi(s,t)](h,k) = [\pi(s)h, \pi(t)k]_H, \quad s, t \in S, \quad h, k \in \mathcal{H}, \tag{7}$$

or if Γ_C is defined by,

$$[C_\pi(s,t)](h,k) = [\pi(t)h, \pi(s)k]_H, \quad s, t \in S, \quad h, k \in \mathcal{H}. \tag{8}$$

Theorem 2.
(i) If π is a pseudo-Hilbert representation of a semigroup and C_π is the kernel of the associated sesquilinear Z-form, then the following assertions take place

(a) The kernel C_π is positive definite;
(b) If π has value in $\mathcal{C}(\mathcal{H})$, $\mathcal{CQ}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ respectively, then the associated kernel C_π satisfies the boundedness conditions (CC), (BCQ) and (BC) respectively.

(ii) If π is a *-pseudo-Hilbert representation of a *-semigroup, then

(a) π is a positive definite operatorial function on a*-semigroup.
(b) If π takes values in $\mathcal{C}^*(\mathcal{H})$, $\mathcal{C}^*\mathcal{Q}(\mathcal{H})$ and $\mathcal{B}^*(\mathcal{H})$ respectively, then for the associated operatorial kernel $\Gamma_\pi(s,t) = \pi(t^* s)$, s, t $\in$ S the boundedness conditions (CC), (BCQ) and (BC) respectively, are satisfied.

Proof: (i) If $\bar{s} = (s_1, ..., s_n) \subset S$, $\bar{h} = (h_1, ..., h_n) \subset \mathcal{H}$, the next calculus

$$\sum_{j,l=1}^{n} C_\pi(s_j, s_l)(h_j, h_l) = \sum_{j,l=1}^{n} [\pi(s_j)h_j, \pi(s_l)h_l] =$$

$$=[\sum_{j=1}^{n} \pi(s_j)h_j, \sum_{l=1}^{n}, \pi(s_l)h_l] \geq 0 \tag{9}$$

or

$$\sum_{j,l=1}^{n} C_\pi(s_j, s_l)(h_l, h_j) = \sum_{j,l=1}^{n} [\pi(s_l)h_l, \pi(s_j)h_j]$$

shows that (a) takes place.

Further on, if we choose $u \in$ S with the above notations we obtain, by an easy calculus, the relation:

$$\sum_{j,l=1}^{n} C_\pi(us_j, us_l)(h_j, h_l) = [\pi(u)\sum_{j=1}^{n} \pi(s_j)h_j, \pi(u)\sum_{l=1}^{n} \pi(s_l)h_l]$$

or

$$\sum_{j,l=1}^{n} C_\pi(us_j, us_l)(h_l, h_j) = [\pi(u)\sum_{l=1}^{n} \pi(s_l)h_l, \pi(u)\sum_{j=1}^{n} \pi(s_j)h_j]$$

hence by applying successively the fact that $\pi(u)$ belongs to $\mathcal{C}(\mathcal{H})$, $\mathcal{CQ}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ respectively, it results that C_π satisfies in an appropriate manner the boundedness conditions (CC), (BCQ) and (BC) respectively.

We should mention that the functions, depending on $p \in \mathcal{P}_Z$ and $u \in$ S which will appear in the boundedness conditions for the kernel C_π are constant, that depend or not of $p \in \mathcal{P}_Z$, depending on the kind of continuity satisfied by the operator $\pi(u)$.

For example, in the last case the function C(u)=$\| \pi(u) \|$ will be used.

(ii) has a similar demonstration, but now, the kernel is operatorial.

3. PROPAGATORS FOR POSITIVE DEFINITE KERNEL ON SEMIGROUPS

Definition 5.

(i) Let S be a multiplicative semigroup without unit and C a $\mathcal{F}(\mathcal{H}$, Z) –valued positive definite kernel on S. A triple ($\mathcal{K}$, D, π) is named minimal propagator of C if $\mathcal{K}$ is a Loynes Z- space, D: $S \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ and π is a pseudo-Hilbert representation of S on $\mathcal{K}$, $\pi : S \to \mathcal{L}(\mathcal{K})$, such that:

$$(\mathcal{K}, D) \text{ is a minimal factorization of } C, \tag{10}$$

$$\pi(t)D(s) = D(ts), \qquad (s, t \in S). \tag{11}$$

(ii) Let S be a *-multiplicative semigroup and C a $\mathcal{F}(\mathcal{H}$, Z)-valued positive definite kernel on S. A triple ($\mathcal{K}$, D, π) is called *-minimal propagator of C if $\mathcal{K}$ is a Loynes Z-space, D: $S \to \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ and π is a *-pseudo-Hilbert representation of S, $\pi : S \to \mathcal{L}^*(\mathcal{K})$ being such that the conditions (10) and (11) to be satisfied.

Lemma 2. Let ($\mathcal{K}$, D, π) be a minimal propagator of the $\mathcal{F}(\mathcal{H}$, Z)-valued kernel C on the *-semigroup S. Then π is a *-representation of S if and only if C has the transfer property (CT).

Proof: If π is a *-representation of S, then for any s, t$\in$ S and h, k$\in$ $\mathcal{H}$, we have:

$$C(us,t)(h,k) = [D(t)h, D(us)k]_{\mathbf{K}} = [\pi(u)^* D(t)h, D(s)k]_{\mathbf{K}} =$$
$$= [D(u^*t)h, D(s)k] = C(s,u^*t)(h,k).$$

Conversely, if the kernel C has the transfer property, then

$$[\pi(t)D(s)h, D(r)h]_{\mathbf{K}} = [D(ts)h, D(r)h]_{\mathbf{K}} = C(r,ts)(h,h) =$$

$$= C(t^*r, s)(h,h) = [D(s)h, D(t^*r)h]_{\mathbf{K}}.$$

We shall deduce that there exists $\pi(t)^*$ and $\pi^*(t)D(r)h = D(t^*r)h,$ (h$\in$ $\mathcal{H}$), $r\in$ S thus $\pi(t)^* \in$ $\mathcal{L}(\mathcal{K})$ by the fact that the space $\mathcal{K}$ is generated by vectors having the form {D(s)h: s$\in$ S, h$\in$ $\mathcal{H}$}.

Now $\pi(t^*)D(r)h = D(t^*r)h$ and $\pi^*(t)D(r)h = D(t^*r)h$ that leads us to $\pi(t^*) = \pi^*(t)$ (t$\in$ S) also by the form of $\mathcal{K}$.

4. DILATIONS FOR $\mathcal{F}(\mathcal{H}$, Z)- VALUED KERNELS ON *-SEMIGROUPS

Next, we shall introduce the notion of minimal dilation (for the Hilbert model, see for example [13]).

Definition 6. The triple ($\mathcal{K}$, R, π) is a minimal dilation (*-dilation) of the kernel C:S×S$\rightarrow$ $\mathcal{F}(\mathcal{H}$, Z) if

$\mathcal{K}$ is a Loynes Z-space, R$\in$ $\mathcal{L}(\mathcal{H}, \mathcal{K})$, (R$\in$ $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$) and π is a representation (12)

(*-representation) of the semigroup (*-semigroup) S, in $\mathcal{L}(\mathcal{K})$ ($\mathcal{L}^*(\mathcal{K})$);

$$C(s,t)(h,k) = [\pi(t)Rh, \pi(s)Rk]_{\mathbf{K}}, \quad (s, t\in S,\ h, k\in \mathcal{H}), \tag{13}$$

(for the case when *-dilation takes the form C(s,t)=R* $\pi(t^*s)R$ (s, t$\in$ S));

$$\mathcal{K} = \vee\{\pi(s)R\ \mathcal{H} : s\in S\}. \tag{14}$$

Two such minimal dilations ($\mathcal{K}$, R, π) and ($\mathcal{K}'$, R', π') are called gramian unitarily equivalents if there exists a gramian unitary operator U$\in$$\mathcal{C}$($\mathcal{K}$, ($\mathcal{K}'$) such that:

$$U\pi(s) = \pi'(s)U, \quad s\in S; \tag{15}$$

$$UR = R'. \tag{16}$$

If the representation π , takes values in the subalgebras $\mathcal{C}(\mathcal{H})$, $\mathcal{CQ}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ respectively, and R is in the subspace $\mathcal{C}(\mathcal{H}, \mathcal{K})$, $\mathcal{CQ}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ respectively, the corresponding dilation $(\mathcal{K}, R, \pi)$ will be named $\mathcal{C}$-, $\mathcal{CQ}$-, $\mathcal{B}$- minimal dilation of C.

Let S be a *-semigroup.

Let

$$[p(\sum_{j=1}^{n} C(s_j,t_j)(h_j,h))]^2 \le C_p(h)p(\sum_{j,l=1}^{n} C(s_j^* t_j, s_l^* t_l)(h_l,h_j)), \tag{17}$$

$(s_1,...,s_n \in S, t_1,...,t_n \in S, h_1,...,h_n, h \in \mathcal{H})$, for any $p \in \mathcal{P}_Z$.

Proposition 2. Let $\mathcal{H}$ be a Loynes Z-space, and S a *-semigroup, and C a $\mathcal{F}(\mathcal{H}, Z)$-valued kernel on S, which satisfies the inequality (17). Then:

(i) there exists a function A : S→ $\mathcal{F}(\mathcal{H}, Z)$ such that

$$C(s,t) = A(s^* t) \quad (s, t \in S), \tag{18}$$

(ii) the kernel C has the transfer property (CT).

Proof: The affirmation (ii) immediately results from (i), because

$$C(us,t) = A(s^* u^* t) = C(s, u^* t).$$

To check (i) we shall define A : S→ $\mathcal{F}(\mathcal{H}, Z)$ by

$$A(u) = \begin{cases} C(s,t), u \in S \cdot S; u = s^* t \\ 0, u \notin S \cdot S. \end{cases}$$

If $u = s_1^* t_1 = s_2^* t_2$ then taking into account that $C(s_1,t_1) = C(s_2,t_2)$ we obtain that A is correctly defined.

Indeed, by nothing k with h_1 and –k with h_2 from (17), for n = 2 it results that

$$[p(C(s_1,t_1)(k,h) - C(s_2,t_2)(k,h))]^2 \le C_p(h)p(\sum_{j,l=1}^{2} C(s_j^* t_j, s_l^* t_l)(h_l,h_j)) =$$

$$= C_p(h)p(\sum_{l,j=1}^{2} C(u,u)(h_l,h_j)) = C_p(h)p(C(u,u)(k,k) - C(u,u)(k,k) -$$

$$- C(u,u)(k,k) + C(u,u)(k,k)) = 0,$$

i.e. $p(C(s_1,t_1)(k,h) - C(s_2,t_2)(k,h)) = 0$

for any $p \in \mathcal{P}_Z$, therefore

$C(s_1,t_1)(k,h) = C(s_2,t_2)(k,h)$ for any h, k∈ $\mathcal{H}$, which completes the proof.

Having in mind that a function A : S→ $\mathcal{F}(\mathcal{H}, Z)$ is positive definite (i.e. it satisfies a boundedness condition respectively) if the kernel $C_A : S \times S \to$ $\mathcal{F}(\mathcal{H}, Z)$ given by $C_A(s,t) = A(t^* s)$ (s, t∈ S) is positive definite (i.e. it respectively satisfies the corresponding boundedness condition from Definition 3), the notion of *-minimal dilation introduced in Definition 6 makes sense for such functions.

It has a natural form. As it can be seen from:

Remark 2. If we consider a *-minimal dilation ($\mathcal{K}$, R, π) (according to Definition 6) of the kernel C_A associated to A, and if we observe that $A(s) = C_A(s,e)$, by applying the relation (13) for t = e, we obtain

$$A(s) = R^*\pi(s)R, \quad s\in S.$$

Therefore, it is justified for the *-minimal dilation ($\mathcal{K}$, R, π) of C_A, to be the *-minimal dilation for the function A. For these functions, the following take place:

Consequence 2.
(i) Let A be a $\mathcal{F}(\mathcal{H}, Z)$-valued function on S. If A satisfies the boundedness conditions (BCQ) and if A is *-dilatable (i.e. has a *-minimal dilation), then there exists a function C_p : $\mathcal{H}\to \mathbf{R}_+$ such that for any $\bar{s} = (s_1,...,s_n)\subset S,\ t = (t_1,...,t_n)\subset S,\ \bar{h} = (h_1,...,h_n)\subset \mathcal{H},\ h\in \mathcal{H}$

$$[p(\sum_{l=1}^{n} A(s_l)(h,h_l))]^2 \le C_p(h)p(\sum_{j,l=1}^{n} A(s_l^*s_j)(h_l,h_j)). \tag{19}$$

(ii) Let S_0 be a sub-semigroup of *-semigroup S such that $S_0^* = S_0$ and A : S$\to$ $\mathcal{F}(\mathcal{H}$, Z) a positive definite function which satisfies the boundedness condition (BCQ). If there exists a Loynes Z-space $\mathcal{K}$, a function D : S$\to$ $\mathcal{F}(\mathcal{H}$, Z), a*-representation π of S on $\mathcal{K}$ and an operator R$\in$ $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$ such that:

($\mathcal{H}$, D, π) is a minimal propagator of C_A
D(s) = π(s)R (s$\in$ S_0)
A(s)(h,k) = $[Rh,\pi(s)Rk]_{\mathbf{K}}$ (s$\in$ S_0, h, k$\in$ $\mathcal{H}$),

then there is a function C_p : $\mathcal{H}\to \mathbf{R}_+$ so that, for any finite sequences $s_1,...,s_n\in S_0$ and $h_1,...,h_n\in \mathcal{H}$, h $\in$ $\mathcal{H}$ we have

$$[p(\sum_{l=1}^{n} A(s_l)(h,h_l))]^2 \le c_p(h)p(\sum_{j,l=1}^{n} A(s_l^*s_j)(h_l,h_j)) \tag{20}$$

for any $p\in \mathcal{P}_Z$.

Proof: (i) Let ($\mathcal{K}$, R, π) be a *-minimal dilation of A. Then

$$[p(\sum_{l=1}^{n} A(s_l)(h,h_l))]^2 = [p(\sum_{l=1}^{n}[Rh,\pi(s_l)Rh_l]_{\mathrm{K}})]^2 =$$
$$= [p([Rh,\sum_{l=1}^{n}\pi(s_l)Rh_l]_{\mathrm{K}})]^2 \le 4p([Rh,Rh])p([\sum_{j=1}^{n}\pi(s_j)Rh_j,\sum_{l=1}^{n}\pi(s_l)Rh_l]_{\mathrm{K}}) =$$
$$= 4q_p^2(Rh)\cdot p(\sum_{j,l=1}^{n}[Rh_j,\pi(s_j^*s_l)Rh_l]_{\mathrm{K}}) = 4q_p^2(Rh)\cdot p(\sum_{j,l=1}^{n} A(s_j^*s_l)(h_j,h_l)),$$

i.e. exactly (19) if we take into account that A satisfies (BCQ) and use the fact that R is in $\mathcal{CQ}(\mathcal{H}, \mathcal{K})$. Therefore, for R $\in$ $\mathcal{CQ}(\mathcal{H}, \mathcal{K})$ we have $p([Rh,Rh]) = q_p^2(Rh) \le M_p^2 p([h,h])$.

Considering $C_p(h) = 4M_p^2 p([h,h])$, we obtain exactly the desired inequality.
(iii) is obvious from (i).

Consequence 3. We notice that for a $\mathcal{G}(\mathcal{H})$- valued positive definite function ϕ, (where $\mathcal{G}$ can be $\mathcal{C}$, $\mathcal{CQ}$ and $\mathcal{B}$ respectively) there is an operator R $\in \mathcal{G}(\mathcal{H}, \mathcal{K})$ (where $\mathcal{G}$ can be $\mathcal{C}$, $\mathcal{CQ}$ and $\mathcal{B}$ respectively) such that

$$\phi(t^* s) = R\pi(t)^* \pi(s)R, \quad t, s \in S,$$

is a *-semigroup with unit.

The following two theorems are analogue of the famous principal theorem of B. Sz.-Nagy [8] for pseudo-Hilbert spaces. We mention that there are many extensions of this result (see A. Weron, F.H. Szafraniec, J. Stochel [9-13]).

Theorem 3. Let Z be an admissible space in the Loynes sense, $\mathcal{H}$ a pre-Loynes Z-space and S a *-semigroup with unit e.

If T is a $\mathcal{L}^*(\mathcal{H})$ – valued positive definite function on S, then there exists a pre-Loynes Z-space $\mathcal{K}_0$, a *-representation π : S$\rightarrow$ $\mathcal{L}^*(\mathcal{K}_0)$ i.e.

$$\pi(e) = I_{\mathbf{K}_0}, \quad \pi(st) = \pi(s)\pi(t), \quad \pi(s)^* = \pi(s^*), \quad s, t \in S \tag{21}$$

and an operator R$\in \mathcal{L}^*(\mathcal{H}, \mathcal{K}_0)$ such that

$$T(s) = R^* \pi(s)R, \quad s \in S. \tag{22}$$

Moreover, $\mathcal{K}_0$ satisfies the minimality condition in the sense that it is algebraically generated by the vectors $\{\pi(s)Rh,\ s\in S,\ h\in \mathcal{H}\}$.

Proof: We consider the $\mathcal{F}(\mathcal{H}, Z)$ – valued kernel $B_{T(s,t)}$, s, t $\in$ S associated with T(. , .) and then the derived kernel $\Gamma_T = \Gamma_{B_T} : \Lambda \times \Lambda \rightarrow Z$ defined by

$$\Gamma_T(\lambda, \mu) = [h, T(s^* t)k], \quad \lambda = (s,h), \mu = (t,k) \in S \times \mathcal{H} = \Lambda.$$

From hypothesis, Γ_T will be a Z-valued kernel positive definite too. We denote by $\mathcal{K}_0$ the pre-Loynes space with the reproducing kernel, determined by the kernel Γ_T.

Using the condition given in Theorem 1, it is known that this $\mathcal{K}_0$ consists of all linear finite combinations of the Z-valued functions defined on $\Lambda = S \times \mathcal{H}$ having the form

$$\{\sum_{j=1}^{n} c_j \Gamma_T(\lambda_j, \cdot), n \in \mathbf{N},\ c_1,\ldots,c_n \in \mathbf{C},\ \lambda_1,\ldots,\lambda_n \in \Lambda\}, \tag{23}$$

and its gramian is defined by:

$$[k_1, k_2]_{\mathbf{K}_0} = \sum_{j,l=1}^{n} c_j^1 c_l^2 \Gamma_T(\lambda_j^1, \lambda_l^2), \quad \text{where} \quad k_\nu = \sum_{j=1}^{n} c_j^\nu \Gamma_T(\lambda_j^\nu, \cdot), \nu = 1,2. \tag{24}$$

Now, we construct the linear operator R$\in \mathcal{L}^*(\mathcal{H}, \mathcal{K}_0)$. For h $\in \mathcal{H}$ we consider $\lambda_h = (e,h) \in \Lambda$ and we define Rh = $\Gamma_T(\lambda_h, \cdot)$, h $\in \mathcal{H}$. An easy calculus shows that R$\in \mathcal{L}(\mathcal{H}, \mathcal{K}_0)$.

Indeed if $\mu = (t,h) \in \Lambda$, then:

$$[R(c_1h_1+c_2h_2)](\mu)=\Gamma_T(\lambda_{c_1h_1+c_2h_2},\mu)=[c_1h_1+c_2h_2,T(e^*t)h]=$$
$$=c_1[h_1,T(e^*t)h]+c_2[h_2,T(e^*t)h]=[c_1\Gamma_T(\lambda_{h_1},\cdot)+c_2\Gamma_T(\lambda_{h_2},\cdot)](\mu)=$$
$$=(c_1Rh_1+c_2Rh_2)(\mu),$$
$$(c_1,c_2\in\mathbf{C},\ h_1,h_2\in\mathcal{H}).$$

For the existence of the adjoint we proceed as below.

We take $k\in\mathcal{K}_0$ arbitrarily, $k=\sum_{j=1}^{n}c_j\Gamma_T(\lambda_j,\cdot)$, $(c_j\in\mathbf{C},\ \lambda_j=(s_j,h_j)\in\Lambda,\ j=1,...,n)$ and for $h\in\mathcal{H}$ applying the condition (24) we obtain successively

$$[Rh,k]_{K_0}=[\Gamma_T(\lambda_h,\cdot),\sum_{j=1}^{n}c_j\Gamma_T(\lambda_j,\cdot)]=\sum_{j=1}^{n}\overline{c}_j\Gamma_T(\lambda_h,\lambda_j)=\sum_{j=1}^{n}\overline{c}_j[h,T(e^*s_j)h_j]\ \mathcal{H}\ =$$
$$=[h,\sum_{j=1}^{n}c_jT(s_j)h_j]_{\mathbf{H}},$$

which show that R* exists and

$$R^*k=\sum_{j=1}^{n}c_jT(s_j)h_j,\qquad k=\sum_{j=1}^{n}c_j\Gamma_T(\lambda_j,\cdot). \tag{25}$$

Now we define the representation π : S$\to$ $\mathcal{L}^*(\mathcal{K}_0)$ by

$$\pi(s)(\sum_{j=1}^{n}c_j\Gamma_T(\lambda_j,\cdot))=\sum_{j=1}^{n}c_j\Gamma_T(\lambda^s{}_j,\cdot),\qquad s\in S \tag{26}$$

where $\lambda_j=(s_j,h_j)\in\Lambda$ and $\lambda^s{}_j=(s^*s_j,h_j)$.

It is obvious that $\pi(s)\in\mathcal{L}(\mathcal{K}_0)$ and it satisfies the first two relations from (21). For the last one we consider $k_\nu=\sum_{j=1}^{n}c_j^\nu\Gamma_T(\lambda_j^\nu,\cdot),\nu=1,2,\ \lambda_j^\nu=(s_j^\nu,h_j^\nu),\ j=1,2,...,n$ and the product

$$[\pi(s)k_1,k_2]_{\mathbf{K_0}}=\sum_{j,l=1}^{n}c_j^1\overline{c_l}^2\Gamma_T(\lambda_j^{1,s},\lambda_l^{2,s})=\sum_{j,l=1}^{n}c_j^1\overline{c_l}^2[h_j^1,T(s_j^{1*}s^*s_l^2)h_l^2]=$$
$$=\sum_{j,l=1}^{n}c_j^1\overline{c_l}^2\Gamma_T(\lambda_j^1,\lambda_l^{2,s^*})=[k_1,k_2^*]_{\mathbf{K_0}},$$

where $k_2^*=\sum_{l=1}^{n}c_l^2\Gamma_T(\lambda_l^{2,s^*})$.

It results that there exists $\pi(s)^*$ and taking this into account and the definition (26), the following takes place

$$\pi(s)^*k_2=k_2^*=\pi(s^*)k_2.$$

Now taking into account the definition of $\pi(s)$ and R, as well as the expression of R* (see the formula (25)) because $\lambda_h^s=(s^*,h)$ we have successively for any $h\in\mathcal{H}$:

$$R^*\pi(s)Rh = R^*\pi(s)\,\Gamma_T(\lambda_h,\cdot) = R^*\Gamma_T(\lambda_h^s,\cdot) = T(s)h,$$

s∈ S, i.e. (22).

Because of $\sum_{j=1}^{n} c_j\Gamma_T(\lambda_j,\cdot) = \sum_{j=1}^{n} c_j\Gamma_T(\lambda_{h_j}^{s_j},\cdot) = \sum_{j=1}^{n} c_j\pi(s_j)Rh_j$ results that the minimality condition takes place.

Theorem 4. Let $\mathcal{H}$ be a Loynes Z-space and S a*-semigroup with unit e. If T is a $\mathcal{C}^*(\mathcal{H})$ – valued positive definite function on *-semigroup S which satisfies the boundedness condition (Definition 2 and 3) respectively:

(BC) there exists a function ρ : S→ [0,∞) such that $\rho(u)B_T - (B_T)_u$ is positive definite for any u ∈ S, i.e.

$$\sum_{i,j=1}^{n} c_i\bar{c}_j[h_i, T(s_i^* u^* u s_j)h_j] \le \rho(u)\sum_{i,j=1}^{n} c_i\bar{c}_j[h_i, T(s_i^* s_j)h_j];$$

(BCQ) Γ_{B_T} satisfies (BCQ) i.e. for any $p \in \mathcal{P}_Z$, there exists a function c_p : S→ [0,∞) such that

$$p(\sum_{i,j=1}^{n} c_i\bar{c}_j[h_i, T(s_i^* u^* u s_j)h_j]) \le c_p(u)p(\sum_{i,j=1}^{n} c_i\bar{c}_j[h_i, T(s_i^* s_j)h_j]);$$

(CC) Γ_{B_T} satisfies (CC) i.e. for any $p \in \mathcal{P}_Z$, there exist $\gamma_p : S \to [0,\infty)$ and c_p:S→[0,∞) so that

$$p(\sum_{i,j=1}^{n} c_i\bar{c}_j[h_i, T(s_i^* u^* u s_j)h_j]) \le c_p(u)\gamma_p(u)(\sum_{i,j=1}^{n} c_i\bar{c}_j[h_i, T(s_i^* s_j)h_j]);$$

for any n ∈ **N**, $c_1,...,c_n \in$ **C**, $s_i, u \in S, h_i \in S, (i=\overline{1,n})$; then there exists a Loynes Z-space $\mathcal{K}$, a *-representation of S in $\mathcal{C}^*(\mathcal{K})$, an operator R∈ $\mathcal{C}^*(\mathcal{H}, \mathcal{K})$ so that a relation like (22) holds.

Moreover, the boundedness conditions (BC), (BCQ) and (CC) of T are transferred to the representation π like this

(B) $[\pi(s)k, \pi(s)k] \le c(s)[k,k]$,

(CQ) $p([\pi(s)k, \pi(s)k]) \le c_p(s)p([k,k])$,

(C) $p([\pi(s)k, \pi(s)k]) \le c_p(s)\gamma_p(s)([k,k])$,

for any s∈S, k∈$\mathcal{K}$, which means that π takes values in $\mathcal{B}^*(\mathcal{K})$, $\mathcal{CQ}^*(\mathcal{K})$ and $\mathcal{C}^*(\mathcal{K})$ respectively.

Proof: Obviously, he first part takes place taking $\mathcal{K}$ as a Loynes Z-space with the reproducing kernel Γ_T, i.e. the functional completion of the pre-Loynes space $\mathcal{K}_0$ from Theorem 3.

Because T is a $\mathcal{C}^*(\mathcal{H})$- valued positive definite function, which satisfies one of the boundedness conditions (BC), (BCQ) and (CC) respectively, T will be $\mathcal{B}^*(\mathcal{H})$-, $\mathcal{CQ}^*(\mathcal{H})$- and $\mathcal{C}^*(\mathcal{H})$- valued respectively. Now, by expressing the gramian [Rh, Rh] we obtain:

$$[Rh, Rh] = [\Gamma_{B_T}(\lambda_h,\cdot), \Gamma_{B_T}(\lambda_h,\cdot)] = \Gamma_{B_T}(\lambda_h,\lambda_h) = [h, T(e)h] \;,$$

$\lambda_h = (e,h)$. Moreover, by applying (22) for s = e we have:

$$[T(e)h, h] = [R^*Rh, h] = [Rh, Rh] \ge 0 \quad \text{for all } h \in \mathcal{H},$$

therefore T(e)∈ $\mathcal{C}_+^*(\mathcal{H})$. When T(s)∈ $\mathcal{B}^*(\mathcal{H})$, we have T(e)∈ $\mathcal{B}_+^*(\mathcal{H})$. It easily results that R∈ $\mathcal{B}^*(\mathcal{H}, \mathcal{K})$. In the other three cases,

$$q_p^{\mathrm{K}}(Rh) = p^2([Rh, Rh]) = p^2([h, T(e)h]) \leq 4q_p^{\mathrm{K}}(h)q_p^{\mathrm{K}}(T(e)h)$$

and using the fact that T(e) is $\mathcal{CQ}^*(\mathcal{H})$- and $\mathcal{C}^*(\mathcal{H})$- valued, we obtain that R is contained in $\mathcal{CQ}^*(\mathcal{H}, \mathcal{K})$ and $\mathcal{C}^*(\mathcal{H}, \mathcal{K})$ respectively, if we also show that R* is continuous with the condition (CC) for T. This fact is obvious from (25).

Now,

$$[\pi(s)k, \pi(s)k] = [\sum_{j=1}^{n} c_j \Gamma_T(\lambda_j^s, \cdot), \sum_{k=1}^{n} c_k \Gamma_T(\lambda_k^s, \cdot)] =$$

$$= \sum_{j,k=1}^{n} c_j \overline{c}_k \Gamma_T(\lambda_j^s, \lambda_k^s) = \sum_{j,k=1}^{n} c_j \overline{c}_k [h_j, T((s^* s_j)^* (s^* s_k)) h_k] =$$

$$= \sum_{j,k=1}^{n} c_j \overline{c}_k [h_j, T(s_j^* s s^* s_k) h_k],$$

and

$$[k, k] = \sum_{j,k=1}^{n} c_j \overline{c}_k \Gamma_T(\lambda_j, \lambda_k) = \sum_{j,k=1}^{n} c_j \overline{c}_k [h_j, T(s_j^* s_k) h_k].$$

Then using the corresponding conditions (BC), (BCQ) and (CC) we obtain:

(B) $[\pi(s)k, \pi(s)k] \leq C(s)[k, k]$, with C(s) = $\rho(s^*)$;

(CQ) $p([\pi(s)k, \pi(s)k]) \leq C_p(s) p([k, k])$, with $C_p(s) = c_p(s^*)$;

(C) $p([\pi(s)k, \pi(s)k]) \leq C_p(s)\gamma'_p(s)([k, k])$, with $\gamma'_p(s) = \gamma_p(s^*)$.

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