ORIGINAL PAPER

TRIGONOMETRICAL IDENTITIES AND INEQUALITIES

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Abstract. *In this paper we present some new identities and inequalities in triangles and quadrilaterals, continuing the idea from [1].*
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1. IDENTITIES AND INEQUALITIES IN TRIANGLES

In the following, let the triangle ABC with the sides $AB = c$, $BC = a$, $CA = b$, the measurements of angles A,B,C, s the semiperimeter, R the circumradius, r the inradius and T the area.

Theorem 1.1. The identity

$$\left((a-b)\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2=c^2 \tag{1.1.}$$

and his permutations hold.

Proof: 1. We have

$$\left((a-b)\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2=$$
$$=(a-b)^2\frac{s(s-c)}{ab}+(a+b)^2\frac{(s-a)(s-b)}{ab}=$$
$$=(a-b)^2\frac{(a+b)^2-c^2}{4ab}+(a+b)^2\frac{c^2-(a-b)^2}{4ab}=$$
$$=\frac{c^2}{4ab}\left[(a+b)^2-(a-b)^2\right],$$

from where (1.1) results.

2. We have

$$\left((a-b)\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2=$$
$$=a^2\cos^2\frac{C}{2}-2ab\cos^2\frac{C}{2}+b^2\cos^2\frac{C}{2}+a^2\sin^2\frac{C}{2}+2ab\sin^2\frac{C}{2}+b^2\sin^2\frac{C}{2}=$$

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$$= a^2 + b^2 - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right) =$$
$$= a^2 + b^2 - 2ab\cos C = c^2$$

so identity (1.1) is true. □

Corollary 1.1. The following statements

1) $C = \frac{\pi}{3}$ *if and only if* $a^2 - ab + b^2 = c^2$

2) $C = \frac{\pi}{5}$ *if and only if* $2\left(a^2 + b^2\right) - \left(\sqrt{5} + 1\right)ab = 2c^2$

3) $C = \frac{\pi}{2}$ *if and only if* $a^2 + b^2 = c^2$

4) $C = \frac{2\pi}{3}$ *if and only if* $a^2 + ab + b^2 = c^2$

hold.

Proof. In the Theorem 1.1 we take $C \in \left\{\frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{3}\right\}$ □

Remark 1.1. The statement 3) from Corollary 1.1 is the well-known Pitagora's theorem.

Corollary 1.2. The inequalities

$$\sin\frac{C}{2} \le \frac{c}{a+b} \tag{1.2}$$

$$|a-b|\cos\frac{C}{2} < c \tag{1.3}$$

and his permutations hold.

Proof. From (1.1) it results that $(a+b)^2\sin^2\frac{C}{2} \le c^2$ and $(a-b)^2\cos^2\frac{C}{2} \le c^2$, from where (1.2) and (1.3) results. In (1.2) the equality holds if and only if $a = b$. In (1.3) we can't have equality because $(a+b)\sin\frac{C}{2} \ne 0$. □

Remark 1.2. The inequality (1.2) appears in [1].

Remark 1.3. If $C = \frac{\pi}{2}$, from (1.2) the following well-known inequality $a + b \le c\sqrt{2}$ results.

Corollary 1.3. We have

$$\frac{r}{4R} \le \frac{2Rr}{s^2 + r^2 + 2Rr} \le \frac{1}{8} \tag{1.4}$$

and

$$\left|(a-b)(b-c)(c-a)\right| < 16R^2 r \tag{1.5}$$

Proof. Using (1.2) we obtain that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{abc}{(a+b)(b+c)(c+a)}$.

But

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R},$$

$$abc=4RT=4srR,$$

$$(a+b)(b+c)(c+a)=(2s-c)(2s-a)(2s-b)=2s\left(s^2+r^2+2Rr\right),$$

$$\frac{abc}{(a+b)(b+c)(c+a)}\le\frac{1}{8}$$

and then (1.4) results.

From (1.3) we have that $\left|(a-b)(b-c)(c-a)\right|<\dfrac{abc}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$ and taking $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}$ into account, inequality (1.5) results. □

Remark 1.4. The inequality (1.4) is a refinement of Euler's $R\ge 2r$ inequality.

Corollary 1.4. The identity

$$\left((a-b)\cos\frac{C}{2}\right)^6+\left((a+b)\sin\frac{C}{2}\right)^6+\frac{3}{4}\left(a^2-b^2\right)^2c^2\sin^2 C=c^6 \tag{1.6}$$

and his permutations hold.

Proof. Taking into account that $x+y+z=0$ then $x^3+y^3+z^3=3xyz$.

Therefore, we take $x=\left((a-b)\cos\frac{C}{2}\right)^2$, $y=\left((a+b)\sin\frac{C}{2}\right)^2$ and $z=-c^2$. □

Theorem 1.2. If $\alpha\ge 1$, then

$$\left(|a-b|\cos\frac{C}{2}\right)^{2\alpha}+\left((a+b)\sin\frac{C}{2}\right)^{2\alpha}\ge 2^{1-\alpha}c^{2\alpha} \tag{1.7}$$

and his permutations hold and if $\alpha\in(0,1)$, then

$$\left(|a-b|\cos\frac{C}{2}\right)^{2\alpha}+\left((a+b)\sin\frac{C}{2}\right)^{2\alpha}\le 2^{1-\alpha}c^{2\alpha} \tag{1.8}$$

and his permutations hold.

Proof. The function $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^\alpha$ is a convex function for $\alpha\ge 1$ and then we have that

$$\left(|a-b|\cos\frac{C}{2}\right)^{2\alpha}+\left((a+b)\sin\frac{C}{2}\right)^{2\alpha}\ge 2\left(\frac{\left(|a-b|\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2}{2}\right)^\alpha.$$

Taking (1.1) into account, we obtain (1.7). If $\alpha\in(0,1)$, the function f is a concave function and the proof is similarly. □

Remark 1.5. The equality in (1.7) or (1.8) holds if and only if $|a-b|\cos\frac{C}{2}=(a+b)\sin\frac{C}{2}$, equivalent with $(a-b)^2\frac{s(s-c)}{ab}=(a+b)^2\frac{s(s-c)}{ab}$, from where $c^2(a^2+b^2)=(a-b)^2(a+b)^2$, so $c=\frac{|a-b|(a+b)}{\sqrt{a^2+b^2}}$.

Corollary 1.5. The following inequality

$$|a-b|\cos\frac{C}{2}+(a+b)\sin\frac{C}{2}\le c\sqrt{2} \tag{1.9}$$

and his permutations hold.

Proof. It results from Theorem 1.2 for $\alpha=\frac{1}{2}$. □

Remark 1.6. The inequality (1.9) appears in [1].

Corollary 1.6. The inequalities

$$|a-b|^{\cos^2\frac{C}{2}}(a+b)^{\sin^2\frac{C}{2}}\le c \tag{1.10}$$

and

$$\left(\cos\frac{C}{2}\right)^{(a-b)^2}\left(\sin\frac{C}{2}\right)^{(a+b)^2}\le\left(\frac{c^2}{2(a^2+b^2)}\right)^{a^2+b^2} \tag{1.11}$$

hold.

Proof. Using the weighted $AM-GM$ inequality, we obtain

$$\frac{c^2}{1}=\frac{\left((a-b)\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2}{\cos^2\frac{C}{2}+\sin^2\frac{C}{2}}\ge\left(|a-b|^{2\cos^2\frac{C}{2}}(a+b)^{2\sin^2\frac{C}{2}}\right)^{\frac{1}{\cos^2\frac{C}{2}+\sin^2\frac{C}{2}}}$$

from where (1.10) results.

Similarly

$$\frac{c^2}{2(a^2+b^2)}=\frac{\left((a-b)\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2}{(a-b)^2+(a+b)^2}\ge\left[\left(\cos\frac{C}{2}\right)^{2(a-b)^2}\left(\sin\frac{C}{2}\right)^{2(a+b)^2}\right]^{\frac{1}{(a-b)^2+(a+b)^2}}$$

from where (1.11) results. □

Corollary 1.7. The following inequalities

$$c^2\ge|a^2-b^2|\sin C, \tag{1.12}$$

$$|a^2-b^2|c+|b^2-c^2|a+|c^2-a^2|b\le 4R(s^2-r^2-4Rr) \tag{1.13}$$

and

$$|(a-b)(b-c)(c-a)|\le\frac{16R^4r}{s^2+r^2+2Rr} \tag{1.14}$$

hold.

Proof. Using the $AM-GM$ inequality, we have that

$$c^2=\left((a-b)\cos\frac{C}{2}\right)^2+\left((a+b)\sin\frac{C}{2}\right)^2\geq 2\sqrt{(a-b)^2(a+b)^2\cos^2\frac{C}{2}\sin^2\frac{C}{2}}=\left|a^2-b^2\right|\sin C,$$

so (1.12). Taking (1.12) into account, we have

$$2\left(s^2-r^2-4R\right)=a^2+b^2+c^2\geq\left|a^2-b^2\right|\sin C+\left|b^2-c^2\right|\sin A+\left|c^2-a^2\right|\sin B=$$

$$=\frac{1}{2R}\left(\left|a^2-b^2\right|c+\left|b^2-c^2\right|a+\left|c^2-a^2\right|b\right)$$

and the inequality (1.13) is proved. Using (1.12) we have

$$16s^2R^2r^2=a^2b^2c^2\geq\left|(a-b)(b-c)(c-a)\right|(a+b)(b+c)(c+a)\sin A\sin B\sin C$$

and taking into account that

$$(a+b)(b+c)(c+a)=2s\left(s^2+r^2+2Rr\right),\ \sin A=\frac{a}{2R},\ \sin B=\frac{b}{2R},\ \sin C=\frac{c}{2R},$$

the inequality (1.14) is obtained. □

Corollary 1.8. The inequality

$$4\sqrt{2}\left(s^2-r^2-4Rr\right)R\leq\sqrt{\left((4R+r)^2-s^2\right)\sum(a-b)^4}+\sqrt{\left(8R^2+r^2-s^2\right)\sum(a+b)^4}\quad(1.15)$$

holds.

Proof. Using the $C-B-S$ inequality, we have

$$2\left(s^2-r^2-4Rr\right)=\sum c^2=\sum(a-b)^2\cos^2\frac{C}{2}+\sum(a+b)^2\sin^2\frac{C}{2}$$

$$\leq\sqrt{\left(\sum(a-b)^4\right)\left(\sum\cos^4\frac{C}{2}\right)}+\sqrt{\left(\sum(a+b)^4\right)\left(\sum\sin^4\frac{C}{2}\right)}=$$

$$=\sqrt{\frac{(4R+r)^2-s^2}{8R^2}\sum(a-b)^4}+\sqrt{\frac{8R^2+r^2-s^2}{8R^2}\sum(a+b)^4}$$

from where the inequality (1.15) results. □

Corollary 1.9. The following inequalities

$$c\leq|a-b|\cos\frac{C}{2}+(a+b)\sin\frac{C}{2},\quad(1.16)$$

$$\sqrt{2}\left(|a-b|\cos\frac{C}{2}+(a+b)\sin\frac{C}{2}\right)\geq c+\sqrt{\left|a^2-b^2\right|\sin C}\quad(1.17)$$

and his permutations hold.

Proof. The identity from (1.1) is equivalent with

$$\left(|a-b|\cos\frac{C}{2}+(a+b)\sin\frac{C}{2}\right)^2-|a-b|(a+b)\sin C=c^2$$

from where (1.16) results. We have that

$$\left(|a-b|\cos\frac{C}{2}+(a+b)\sin\frac{C}{2}\right)^2=c^2+\left|a^2-b^2\right|\sin C\geq\frac{1}{2}\left(c+\sqrt{\left|a^2-b^2\right|\sin C}\right)^2$$

from where we obtain (1.17). □

Corollary 1.10. The inequalities

$$2s \le \sum 2|a-b|\cos\frac{C}{2} + \sum (a+b)\sin\frac{C}{2} \le \min\left\{2\sqrt{2}s, 2\sqrt{3\left(s^2-r^2-4Rr\right)}\right\} \tag{1.18}$$

hold.

Proof. Taking Corollary 1.5 into account we have that

$$\sum |a-b|\cos\frac{C}{2} + \sum (a+b)\sin\frac{C}{2} \le \sum c\sqrt{2} = 2\sqrt{2}s.$$

In another order, we have that

$$\sum |a-b|\cos\frac{C}{2} + \sum (a+b)\sin\frac{C}{2} \le 6\sqrt{\frac{1}{6}\left(\sum\left(|a-b|\cos\frac{C}{2}\right)^2 + \left((a+b)\sin\frac{C}{2}\right)^2\right)} =$$

$$= 6\sqrt{\frac{1}{6}\sum c^2} = 2\sqrt{3\left(s^2-r^2-4Rr\right)}.$$

From the inequalities (1.18) demonstrated above, the second inequality from (1.18) is true.

The first inequality from (1.18) results immediately from the (1.16) inequality. □

Corollary 1.11. The following inequalities

$$\sqrt{\sum\left(|a-b|\cos\frac{C}{2} + (a+b)\sin\frac{C}{2}\right)^2} \le \sqrt{\sum\left((a-b)\cos\frac{C}{2}\right)^2} + \sqrt{\sum\left((a+b)\sin\frac{C}{2}\right)^2} \le 2\sqrt{s^2-r^2-4Rr} \tag{1.19}$$

are true.

Proof. Using the Minkowski inequality and $|x+y| \le \sqrt{2\left(x^2+y^2\right)}$ inequality, we have that

$$\sqrt{\sum\left(|a-b|\cos\frac{C}{2} + (a+b)\sin\frac{C}{2}\right)^2} \le \sqrt{\sum\left((a-b)\cos\frac{C}{2}\right)^2} + \sqrt{\sum\left((a+b)\sin\frac{C}{2}\right)^2}$$

$$\le \sqrt{2\sum\left(\left((a-b)\cos\frac{C}{2}\right)^2 + \left((a+b)\sin\frac{C}{2}\right)^2\right)} = \sqrt{2\sum c^2},$$

from where inequality (1.19) results. □

2. IDENTITIES AND INEQUALITIES IN QUADRILATERALS

In the following, let the convex quadrilateral *ABCD* with the sides *AB* = *a*, *BC* = *b*, *CD* = *c*, *DA* = *d*, the diagonals *BD* = *e, AC* = *f*, the measure of angels equal with *A, B, C,* respectively *D, s, R, r, T* denote the semiperimeter, circumradius, inradius and area.

Theorem 2.1. The identity

$$\left((a-d)\cos\frac{A}{2}\right)^2 + \left((a+d)\sin\frac{A}{2}\right)^2 = \left((b-c)\cos\frac{C}{2}\right)^2 + \left((b+c)\sin\frac{C}{2}\right)^2 \tag{2.1}$$

and his permutations hold.

Proof. Applying Theorem 1.1 in the triangles *ABC* and *BCD*, we have that

$$BD^2=\left(\left(a-d\right)\cos\frac{A}{2}\right)^2+\left(\left(a+d\right)\sin\frac{A}{2}\right)^2$$

and

$$BD^2=\left(\left(b-c\right)\cos\frac{C}{2}\right)^2+\left(\left(b+c\right)\sin\frac{C}{2}\right)^2$$

from where the identity (2.1) results. □

Corollary 2.1. If ABCD is a cyclic quadrilateral, then the identities

$$\left(\left(a-d\right)\cos\frac{A}{2}\right)^2+\left(\left(a+d\right)\sin\frac{A}{2}\right)^2=\left(\left(b-c\right)\sin\frac{A}{2}\right)^2+\left(\left(b+c\right)\cos\frac{A}{2}\right)^2, \qquad (2.2)$$

$$\sin^2\frac{A}{2}=\frac{(b+c)^2-(a-d)^2}{4(ad+bc)} \qquad (2.3)$$

and his permutations hold.
Proof. In Theorem 2.1 we take $C=\pi-A$. □

Corolarry 2.2. Let ABCD be a cyclic quadrilateral. Then

1. $A=\frac{\pi}{2}$ *if and only if* $a^2+d^2=b^2+c^2$;
2. $A=\frac{\pi}{3}$ *if and only if* $a^2+d^2-ad=b^2+c^2+bc$;
3. $A=\frac{\pi}{5}$ *if and only if* $2\left(a^2+d^2\right)-\left(\sqrt{5}+1\right)ad=2(b^2+c^2)+\left(\sqrt{5}+1\right)bc$

and

4. $A=\frac{2\pi}{3}$ *if and only if* $a^2+d^2+ad=b^2+c^2-bc$.

Proof. It results from relation (2.1). □

Corollary 2.3. If ABCD is a cyclic quadrilateral, then

$$(ac+bd)^2=\left(\left(\left(a-d\right)\cos\frac{A}{2}\right)^2+\left(\left(a+d\right)\sin\frac{A}{2}\right)^2\right)\cdot\left(\left(\left(c-d\right)\cos\frac{D}{2}\right)^2+\left(\left(c+d\right)\sin\frac{D}{2}\right)^2\right) \qquad (2.4)$$

and

$$\left(\frac{ab+cd}{ad+bc}\right)^2=\frac{\left(\left(a-d\right)\cos\frac{A}{2}\right)^2+\left(\left(a+d\right)\sin\frac{A}{2}\right)^2}{\left(\left(c-d\right)\cos\frac{D}{2}\right)^2+\left(\left(c+d\right)\sin\frac{D}{2}\right)^2}. \qquad (2.5)$$

Proof. We have

$$BD^2=\left(\left(a-d\right)\cos\frac{A}{2}\right)^2+\left(\left(a+d\right)\sin\frac{A}{2}\right)^2$$

and

$$AC^2=\left(\left(c-d\right)\cos\frac{D}{2}\right)^2+\left(\left(c+d\right)\sin\frac{D}{2}\right)^2.$$

Now, using the Ptolemy's first and second theorem, we obtain the relations (2.4) and (2.5). □

Theorem 2.2. The following inequalities

$$e \geq \max\left\{(a+d)\sin\frac{A}{2}, (b+c)\sin\frac{C}{2}\right\}, \tag{2.6}$$

with the equality if and only if $a = d$ or $b = c$,

$$f \geq \max\left\{(a+b)\sin\frac{B}{2}, (c+d)\sin\frac{D}{2}\right\}, \tag{2.7}$$

with the equality if and only if $a = b$ or $c = d$,

$$e > \max\left\{|a-d|\cos\frac{A}{2}, |b-c|\cos\frac{C}{2}\right\} \tag{2.8}$$

and

$$f > \max\left\{|a-b|\cos\frac{B}{2}, |c-d|\cos\frac{D}{2}\right\} \tag{2.9}$$

hold.

Proof. It results from the proof of Theorem 2.1. □

Theorem 2.3. If $\alpha \geq 1$, then

$$\left(|a-d|\cos\frac{A}{2}\right)^{2\alpha} + \left((a+d)\sin\frac{A}{2}\right)^{2\alpha} + \left(|b-c|\cos\frac{C}{2}\right)^{2\alpha} + \left((b+c)\sin\frac{C}{2}\right)^{2\alpha} \geq 2^{2-\alpha}e^{2\alpha} \tag{2.10}$$

and similarly for *f*, and if $\alpha \in (0,1)$ then

$$\left(|a-d|\cos\frac{A}{2}\right)^{2\alpha} + \left((a+d)\sin\frac{A}{2}\right)^{2\alpha} + \left(|b-c|\cos\frac{C}{2}\right)^{2\alpha} + \left((b+c)\sin\frac{C}{2}\right)^{2\alpha} \leq 2^{2-\alpha}e^{2\alpha} \tag{2.11}$$

and similarly for *f*.

Proof. See the proof of Theorem 1.2. □

Corollary 2.4. The following inequalities

$$|a-d|\cos\frac{A}{2} + (a+d)\sin\frac{A}{2} + |b-c|\cos\frac{C}{2} + (b+c)\sin\frac{C}{2} \leq 2e\sqrt{2} \tag{2.12}$$

and

$$|a-b|\cos\frac{B}{2} + (a+b)\sin\frac{B}{2} + |c-d|\cos\frac{D}{2} + (c+d)\sin\frac{D}{2} \leq 2f\sqrt{2} \tag{2.13}$$

hold.

Proof. It results from Theorem 2.3 for $\alpha = \frac{1}{2}$. □

We recall the following well-known identities from cyclic and tangential quadrilaterals

$$ef = 2r(\sqrt{4R^2 + r^2} + r), \tag{2.14}$$

$$\sum a = 2s, \tag{2.15}$$

$$\sum abc = sef, \tag{2.16}$$

$$abcd = r^2s^2 = T^2, \tag{2.17}$$

$$(ab+cd)(ad+bc) = s^2\left(ef - 4r^2\right), \tag{2.18}$$

$$R^2=\frac{(ab+cd)(ad+bc)ef}{16r^2s^2}, \tag{2.19}$$

(see [5] and [6]).

In the following we consider that *ABCD* is a cyclic and tangential quadrilateral.

Lemma 2.1. The identity

$$\sin\frac{A}{2}=\sqrt{\frac{bc}{ad+bc}} \tag{2.20}$$

and his permutations hold.

Proof. From (2.3) we have that $\sin^2\frac{A}{2}=\frac{(b+c-a+d)(b+c+a-d)}{4(ad+bc)}$ and taking into account that $a+c=b+d$, relation (2.20) results. □

Lemma 2.2. The following identity

$$(a+b)(b+c)(c+d)(d+a)=4r\left[2r\left(2R^2+r^2\right)+(s^2+2r^2)\sqrt{4R^2+r^2}\right] \tag{2.21}$$

holds.

Proof. On verify immediately that

$$(a+b)(b+c)(c+d)(d+a)=\left(\sum a\right)\left(\sum abc\right)+(ac+bd)^2-4abcd$$

and taking into account that $ac+bd=ef$ and the relations (2.14)-(2.16), the identity (2.21) results. □

Theorem 2.4. The following inequalities

$$\left(\frac{R}{r}\right)^2\geq\frac{\sqrt{2\left[\left(4R^2-s^2\right)r\sqrt{4R^2+r^2}+4R^2\left(r^2+s^2\right)+r^2s^2\right]}}{8r^2}\geq 2 \tag{2.22}$$

hold.

Proof. From (2.19), (2.6), (2.7), (2.17) and (2.20) we have

$$\frac{R^2}{r^2}=\frac{(ab+cd)(ad+bc)\sqrt{e^2s^2}}{16r^4s^2}$$

$$\geq\frac{(ab+cd)(ad+bc)\sqrt{(a+d)\sin\frac{A}{2}(b+c)\sin\frac{C}{2}(a+b)\sin\frac{B}{2}(c+d)\sin\frac{D}{2}}}{16r^4s^2}$$

$$=\frac{(ab+cd)(ad+bc)\sqrt{(a+b)(b+c)(c+d)(d+a)\frac{abcd}{(ab+cd)(ad+bc)}}}{16r^4s^2},$$

so

$$\left(\frac{R}{r}\right)^2=\frac{\sqrt{(ab+cd)(ad+bc)(a+b)(b+c)(c+d)(d+a)}}{16r^3s}. \tag{2.23}$$

Taking (2.14)-(2.19) and (2.21) into account, from (2.23) we have

$$\left(\frac{R}{r}\right)^2\geq\frac{\sqrt{s^2 2r\left(\sqrt{4R^2+r^2}-r\right)4r\left[2r\left(2R^2+r^2\right)+\left(s^2+2r^2\right)\sqrt{4R^2+r^2}\right]}}{16r^3s},$$

from where the first inequality from (2.22) results.

On the other hand, from (2.23) we have that

$$\frac{\sqrt{(ab+cd)(ad+bc)(a+b)(b+c)(c+d)(d+a)}}{16r^3s}=$$
$$=\frac{s^2\sqrt{(ab+cd)(ad+bc)(a+b)(b+c)(c+d)(d+a)}}{16\left(\sqrt{abcd}\right)^3}. \tag{2.24}$$

But

$$p^2\geq 4\sqrt{abcd},$$
$$ab+cd\geq 2\sqrt{abcd},$$
$$ad+bc\geq 2\sqrt{abcd},$$
$$(a+b)(b+c)(c+d)(d+a)\geq 16abcd$$

and then, from (2.24) we obtain the second inequality from (2.22). □

Corollary 2.5. (L. Fejes Tóth inequality) If ABCD is a cyclic and tangential quadrilateral, then

$$R\geq r\sqrt{2}. \tag{2.25}$$

Proof. This inequality it results from Theorem 2.4, from the inequality $\left(\frac{R}{r}\right)^2\geq 2$. The equality holds if and only if *ABCD* is a square. □

Remark 2.1. The inequality (2.22) is a refinement of L. Fejes Tóth's $R\geq r\sqrt{2}$ inequality.

Corollary 2.6. The following inequality

$$\left|(a-b)(b-c)(c-d)(d-a)\right|<32R^2r\left(\sqrt{4R^2+r^2}+r\right) \tag{2.26}$$

holds.

Proof. From the relation (2.20) it results immediately that

$$\cos\frac{A}{2}=\sqrt{\frac{ad}{ad+bc}} \tag{2.27}$$

and his permutations. From (2.8), (2.9) and taking (2.27) into account, we have that

$$e^2f^2>|a-d|\cos\frac{A}{2}|b-c|\cos\frac{C}{2}|a-b|\cos\frac{B}{2}|c-d|\cos\frac{D}{2}$$
$$=\left|(a-b)(b-c)(c-d)(d-a)\right|\frac{abcd}{(ab+cd)(ad+bc)}.$$

Using the relations (2.14) and (2.18), the inequality above becomes

$$\left|(a-b)(b-c)(c-d)(d-a)\right|<\frac{4R^2\left(\sqrt{4R^2+r^2}+r\right)^2\cdot s^2\cdot 2r\left(\sqrt{4R^2+r^2}-r\right)}{r^2s^2},$$

from where the inequality (2.26) results. □

Lemma 2.3. The following

$$\sin A\sin B=\frac{r\left(\sqrt{4R^2+r^2}+r\right)}{2R^2}\geq\frac{2r^2}{R^2} \tag{2.28}$$

holds.

Proof. We have $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ and taking (2.20), (2.27), (2.17) into account we obtain that $\sin A = \frac{2T}{ad+bc}$ and similarly $\sin B = \frac{2T}{ab+cd}$. Then $\sin A\sin B = \frac{4T^2}{(ab+cd)(ad+bc)}$ and taking (2.18), (2.14) and (2.17) into account, the identity from (2.28) results. Using the Fejes Tóth inequality, the inequality from (2.28) results. □

Theorem 2.5. The following inequalities

$$\frac{r\sqrt{2}}{R} \le \frac{\sqrt{2}}{4R}\left(\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}+2r\right) \le \frac{1}{2}\left(\cos\frac{A-B}{2}+\frac{r\sqrt{2}}{R}\right) \le$$
$$\le \frac{1}{2}\left(\cos\frac{A-B}{2}+\frac{\sqrt{2r\left(\sqrt{4R^2+r^2}+r\right)}}{2R}\right)$$
$$\le \frac{1}{4}\left(\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-D}{2}+\cos\frac{D-A}{2}\right) \le$$
$$\le \frac{2+\sin A+\sin B}{4} \le 1 \tag{2.29}$$

hold.

Proof. We note $E = \frac{1}{4}\left(\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-D}{2}+\cos\frac{D-A}{2}\right)$.

Then taking $A+C=B+D=\pi$ into account, we have

$$E = \frac{1}{4}(\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+$$
$$+\cos\frac{C}{2}\cos\frac{D}{2}+\sin\frac{C}{2}\sin\frac{D}{2}+\cos\frac{D}{2}\cos\frac{A}{2}+\sin\frac{D}{2}\sin\frac{A}{2}) =$$
$$= \frac{1}{2}\left(\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\sin\frac{A}{2}+\sin\frac{B}{2}\cos\frac{A}{2}\right) =$$
$$= \frac{1}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right).$$

But
$$E \le \frac{\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2+\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right)^2}{4} = \frac{2+\sin A+\sin B}{4},$$

so
$$E \le \frac{2+\sin A+\sin B}{4} \le 1. \tag{2.30}$$

On the other hand,

$$E \ge \frac{1}{2}\left(\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}}+\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}}\right)^2 = \frac{1}{2}\left(\cos\frac{A-B}{2}+\sqrt{\sin A\sin B}\right)$$

and using the relation (2.28), we obtain that

$$E \ge \frac{1}{2}\left(\cos\frac{A-B}{2}+\frac{\sqrt{2r\left(\sqrt{4R^2+r^2}+r\right)}}{2R}\right) \ge \frac{1}{2}\left(\cos\frac{A-B}{2}+\frac{r\sqrt{2}}{R}\right). \tag{2.31}$$

But

$$\cos\frac{A-B}{2}=\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\geq 2\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{A}{2}\sin\frac{B}{2}}=\sqrt{\sin A\sin B}$$

and taking (2.28) into account, we obtain that

$$\cos\frac{A-B}{2}\geq\frac{\sqrt{2r\left(\sqrt{4R^2+r^2}+r\right)}}{2R}\geq\frac{r\sqrt{2}}{R}. \tag{2.32}$$

From (2.30)-(2.32), it results the inequalities from (2.29). □

REFERENCES

[1] Bencze, M., *Octogon Mathematical Magazine*, **15**(2A), 818, 2007.
[2] F. G.-M., *Exercices de géométrie*, Huitième edition, Paris, 1931.
[3] Hadamard, J., *Lectii de geometrie elementara*, Vol. I, II, Ed. Tehnica, Bucuresti, 1960-1961.
[4] Nicolescu, L, Boskoff, V., *Probleme practice de geometrie*, Ed. Tehnica, Bucuresti, 1990.
[5] Pop, O.T., *Gazeta Matematica*, **5-6**, 203, 1988.
[6] Pop, O.T., *Gazeta Matematica*, **8**, 279, 1989.